Wrinkling of Stretched Elastic Films via Global Parametric Bifurcation

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Summary We study the wrinkling formation of a stretched, rectangular thin film with two opposing fixed ends and two opposing free boundaries. We employ the well-known Föppl-von Kármán model, clearly highlighting that the in-plane membrane energy is two orders of magnitude larger than that due to bending. We study the problem as one of perfect bifurcation of the wrinkled state from the planar stretched state. Our point of departure is the use of a novel and powerful method introduced previously in the context of phase transitions in the presence of very small interfacial energy [1]. In essence we use the reciprocal of the squared thickness of the plate as a global bifurcation parameter, avoiding the numerical difficulties associated with extremely thin films.

INTRODUCTION

We study here the specific problem first illustrated in papers [2], [3] by Cerda, et. al., viz., the formation of wrinkles in a stretched rectangular film, as shown below from an experiment carried out at Cornell University:

As in [3] we adopt the Föppl-von Kármán model, but, in contrast, we consider realistic aspect ratios (width-to-length ~ O(1)) in the presence of very small thickness. As such, we are not led to the asymptotic study of an ode problem associated with a beam on an elastic foundation. In particular our 2-dimensional, planar, unwrinkled state is not homogeneous and must be determined numerically. Similar to the pure finite-element study of this problem carried out in [4], we treat wrinkling as a bifurcation phenomenon. In contrast, we do not attempt to approximate and/or guess the correct imperfection near the highly singular, thin-stretched-membrane problem. Rather we fix the global stretch and then study the perfect bifurcation problem associated with the reciprocal of the squared thickness of the thin film, denoted by κ, as the control parameter. By globally following the *stable* branch (local energy minimizers) in κ, we systematically and efficiently determine the physically relevant wrinkled states for extremely small film thicknesses.

FORMULATION

With $\Omega = (0, L) \times (0, W)$ representing the geometry of the undeformed sheet, the total (normalized) potential energy can be expressed by

$$V = \int_{\Omega} \{[(1-\nu)\mathbf{E}_b \cdot \mathbf{E}_b + \nu(tr\mathbf{E}_b)^2] + 12\kappa[(1-\nu)\mathbf{E}_m \cdot \mathbf{E}_m + \nu(tr\mathbf{E}_m)^2]\} dA, \tag{1}$$

where $\mathbf{E}_m \equiv (\varepsilon \mathbf{e}_1 \otimes \mathbf{e}_1 + \nabla\mathbf{u} + \nabla\mathbf{u}^T + \nabla w \otimes \nabla w)/2$ (membrane strain), $\mathbf{E}_b \equiv -\nabla^2 w$ (bending strain),

$\mathbf{u}(x_1, x_2) = u_1(x_1, x_2)\mathbf{e}_1 + u_2(x_1, x_2)\mathbf{e}_2$ (in-plane displ.), $\mathbf{d}(x_1, x_2) = \mathbf{u}(x_1, x_2) + w(x_1, x_2)\mathbf{e}_2$ (total displ.), $(x_1, x_2) \in \Omega$,

ν = Poisson's ratio, $\kappa = 1/h^2$, h = film thickness, $\varepsilon > 0$, imposed global strain.

The geometric (homogeneous) boundary conditions are then given by

$$\mathbf{d}(0, x_2) = \mathbf{d}(L, x_2) = 0, \ 0 \le x_2 \le W. \tag{2}$$

COMPTATIONAL STRATEGY AND RESULTS

We first consider the problem with data $L = 20$, $W = 10$, $\nu = 0.45$ and $\varepsilon = 0.1$. In view of the rectangular symmetry of our problem, we may conveniently compute nonlinear solution branches for 2 reduced problems via the domain $\Omega_r = (0, L/2) \times (0, W/2)$ with reflection symmetry about the line $x_1 = L/2$, and (i) reflection symmetry along $x_2 = W/2$; (ii) rotational symmetry through the angle π radians about the line $x_2 = W/2$. We employ 4-node, rectangular finite elements, with nodal variables $\mathbf{d}$, ∇w and $\partial^2 w / \partial x_1 \partial x_2$ (six degrees of freedom per node). The first

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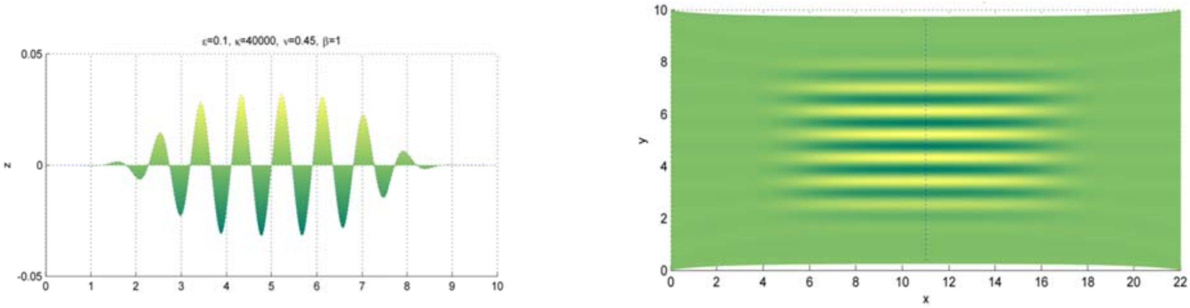
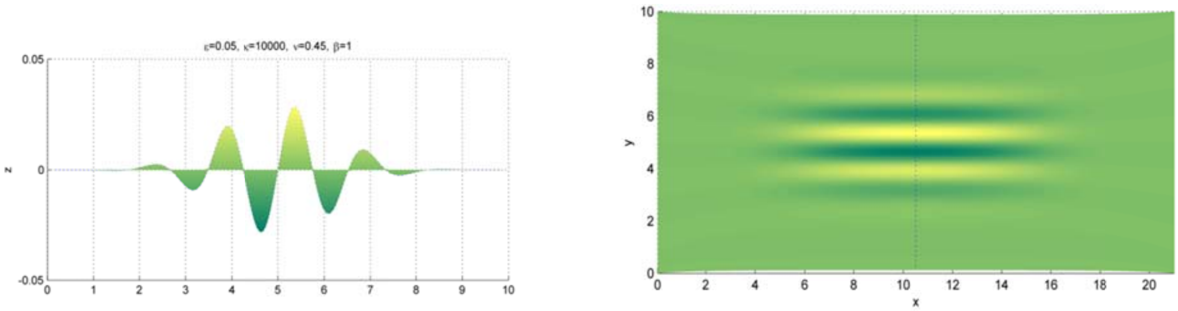
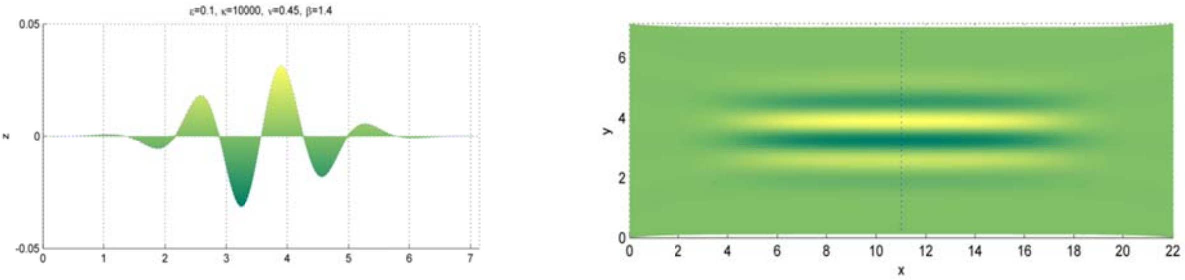
variation of the energy yields the weak form of the equilibrium equations, in which the bifurcation parameter κ appears linearly. For a given boundary value problem (cf. (i), (ii) above), we first compute the planar "trivial" solution, corresponding to $w \equiv 0$; this is a linear problem. We then look for values of $\kappa > 0$ for which the tangent stiffness or Hessian matrix (about the trivial solution) yields zero eigenvalues, i.e, we pinpoint bifurcation points. We then follow, via numerical continuation, the imperfection-free global solutions branches. We check for stability by assembling and monitoring the positivity of the tangent stiffness matrix for the entire structure. *In all cases considered, we find that the stable wrinkled patterns are odd with respect to the line* $x_2 = W/2$ (case (ii) above). Once stable global patterns are found, we then freeze the very small thickness and perform numerical continuation in the stretch $\varepsilon > 0$ and the aspect ratio $\beta \equiv L/2W$. A sample of our results is given below. We consistently used a 70 x 170 grid on the one-quarter domain Ω_r; the refinement in the x_2 direction is necessary to accommodate the high oscillations of the wrinkles.



Stable wrinkling pattern at $\kappa = 40{,}000$, $\nu = 0.45$, $\beta = 1$ and $\varepsilon = 0.1$



Stable wrinkling pattern at $\kappa = 10{,}000$, $\nu = 0.45$, $\beta = 1$ and $\varepsilon = 0.05$



Stable wrinkling pattern at $\kappa = 10{,}000$, $\nu = 0.45$, $\beta = 1.4$ and $\varepsilon = 0.1$

CONCLUSIONS

By using the reciprocal of the small singular-perturbation parameter – in this case the thickness squared of the film – as a global bifurcation parameter, we are able to efficiently and systematically compute stable wrinkled patterns of extremely thin and highly stretched elastic films. The method has a constructive mathematical underpinning [1], illuminating the success of the numerical approach. Indeed, the simple trick of dividing the governing equations through by the small parameter leaves the strongly elliptic, principle part of the operator unmolested, while the lower-order terms take on the role of large forcing. This enables not only sharp existence results (via global bifurcation theory) but systematic numerical approaches as well.

References

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