A NEW APPROACH FOR SOLUTION OF BUCKLING OF FUNCTIONALLY GRADED MATERIAL TIMOSHENKO BEAMS

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Summary Relationships between the solutions of buckling of functionally graded material (FGM) Timoshenko beams and those of the homogenous Euler-Bernoulli beams were investigated. Based on the first-order shear deformation beam theory, governing equation of buckling of FGM beams were derived in terms of the deflection, which has the same form as that of the homogenous Euler-Bernoulli beams. By solving the governing equation with specific boundary conditions, the critical buckling loads of the FGM Timoshenko beams are expressed in terms of those of the corresponding homogenous Euler-Bernoulli beams.

INTRODUCTION

Functionally graded materials (FGM) are inhomogenous composites characterized by a smooth and continuous variation of material in both compositional profile and material properties in space which have found wide application in modern technology and engineering. Consequently, studies devoted to understand the static and dynamic behaviors of FGM beams, plates and shells have being paid more attentions in recent years. However, comparing with the studies on the static and dynamic response on FGM plates and shells, research attentions to the behaviors of FGM beams have relatively limited. Benatta et al.[1], Sallai et al.[2] and Kadoli et al[3] studied static bending of FGM beams based on higher-order shear deformation beam theory by analytical and numerical approaches. Li and Huang[4,5] investigated static bending, buckling and transverse vibration of FGM Timoshenko beams by introducing a new function to decouple the governing equations. Ke et al[6] studied free flexural vibration and buckling of FGM Timoshenko beams containing open cracks. Sankar[7], Zhong and Yu[8], and Ding et al[9] studied transverse bending of FGM beams in the sense of elasticity theory. Different from the conventional approaches, some authors focused on relationship between the bending, buckling and free vibrations of FGM structures and those of homogenous ones. By examining the numerical results in the available literature, Abrate[10,11] found that the static and dynamic responses of a FGM plate are approximately proportional to those of the homogenous counterparts and that the proportionality factors can be determined by the material gradient. By using physical neutral surface, Zhang and Zhou[12] presented solutions of buckling load and frequency of thin FGM plates in terms of the that of the homogenous ones. For homogenous beams, Reddy et al.[13] obtained bending solutions based on the third-order shear deformation theory in terms of the classical solutions for different boundary conditions. In the sense of Euler-Bernoulli theory, Li and Liu[14] proved that deflections, critical loads and natural frequencies of FGM beams are proportional to those of the corresponding homogenous beams and that the proportional coefficients only depend on the material gradients. Furthermore, in this study, we will search for the relationship between static buckling loads of FGM Timoshenko beams and those of the corresponding homogenous Euler-Bernoulli beams. Finally, we want to transform solution of bucking of FGM Timoshenko beams to the calculation of related transition coefficients determined by the regulation of the material gradient and the geometry of the FGM beams if the solutions of the corresponding homogenous Euler-Bernoulli beams are known.

THE GOVERNING EQUATUONS

Consider a uniform beam made from functionally graded materials, having rectangular cross section, with length l, width b and depth h. The material properties are assumed to be varied continuously in the thickness direction from the top to the bottom surfaces. Based on the first-order shear deformation beam theory, governing equations of the buckling of FGM beams can be expressed in terms of displacements in dimensionless form as following:

$$\frac{c_s}{c}\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2}=\frac{\mathrm{d}W}{\mathrm{d}\xi}+\varphi\,,\quad \frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}+\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}=c_sP\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2}\,, \tag{1a,b}$$

in which the dimensionless quantities are defined by

$$(\xi,W,\delta)=(x,w_0,h)/l\,,\quad P=pl^2/(E_bI)\,,\quad c=(\phi_3-12\phi_2^{\,2}/\phi_1)^{-1}\,,\quad c_s=(1+\nu)\delta^2/(6\kappa\phi_1)\,. \tag{2}$$

where x is the axial coordinate taking along the geometrically central axis; z is the transverse coordinate initiated at the geometric middle surface and being positive upward; $w_0(x)$ is the deflection, $\varphi(x)$ is the rotational angle of the cross-section; p is the axial compressive force applied at the ends; E_b is the Young's modulus of the bottom surface; ν is the Poisson ratio which is assumed to be constant in the beam; κ is the Timoshenko shear correct coefficient which equals $5/6$ for the rectangular cross section; I is inertia moment of the cross-section; $\phi_i\,(i=1,2,3)$ are three non-dimensional coefficients determined by the gradient law of the beam materials and determined by

$$\phi_1 = \frac{1}{AE_b}\int_A E(z)\mathrm{d}A,\ \ \phi_2 = \frac{1}{AhE_b}\int_A E(z)z\mathrm{d}A,\ \ \phi_3 = \frac{1}{IE_b}\int_A E(z)z^2\mathrm{d}A$$

where A is the area of the cross-section; $E(z)$ is the Young's modulus varying in the thickness with $E_b = E(-h/2)$.

RELATIONSHIP BETWEEN CRITICAL LODS

Using Eq.(1) we can expressed the rotational angle and its derivative in terms of the deflections as following:

$$\varphi = -\frac{\mathrm{d}W}{\mathrm{d}\xi} - \alpha\frac{\mathrm{d}^3W}{\mathrm{d}\xi^3},\ \ \frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = -(1-c_sP)\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2} \tag{3,a.b}$$

in which $\alpha = c_s(1-c_sP)/c$. Furthermore, we arrive at governing equation only in terms of the deflection following:

$$\frac{\mathrm{d}^4W}{\mathrm{d}\xi^4} + \Lambda^2\frac{\mathrm{d}^2W}{\mathrm{d}\xi^2} = 0, \tag{4}$$

where $\Lambda = \sqrt{cP/(1-c_sP)}$. Obviously, Eq.(4) reduces to the governing equation of Euler-Bernoulli beams in the special case that $c_s = 0$. Especially, for $c = 1$ and $c_s = 0$ it governs the buckling of homogenous Euler-Bernoulli beams. It is easy to find general solution of differential equation (4) as following:

$$W(\xi) = \beta_1\cos\Lambda\xi + \beta_2\sin\Lambda\xi + \beta_3\xi + \beta_4 \tag{5}$$

where $\beta_i\,(i = 1,2,3,4)$ are arbitrary constants to be determined by the specified boundary conditions.
Herein, we consider three common end supports, namely simply supported (S), clamped (C) and free (F). Using Eq.(3) we can denote the boundary conditions of Timoshenko beams only in terms of the deflection, W. Then, for the beams with the combination of the above mentioned three end constraints we can obtain the minimum eigenvalues for Λ. Finally, we can arrive at critical buckling loads of FGM Timoshenko beams. Comparing the dimensionless critical loads of FGM Timoshenko beams with those of homogenous Euler-Bernoulli beams with the same end supports, we prove that for S-S, C-C and C-F end constrains dimensionless critical load parameters of the FGM Timoshenko beams can be expressed in terms of that of Euler-Bernoulli beams as following:

$$P_{\mathrm{Tcr}} = P^*_{\mathrm{E\,cr}}/(c + c_sP^*_{\mathrm{E\,cr}}),\quad \text{or}\quad P_{\mathrm{Tcr}} = P_{\mathrm{Ecr}}/(1 + c_sP_{\mathrm{Ecr}}) \tag{6}$$

where $P^*_{\mathrm{E\,cr}}$ and $P_{E\,\mathrm{cr}}$ are the dimensionless critical loads of the homogenous and FGM Euler-Bernoulli beams, respectively; P_{Tcr} denotes the dimensionless critical load parameter of FGM Timoshenko beams. Unfortunately, for the beams with C-S end constraints, the above mentioned relationship is not valid. However, a simple eigenvalue equation is derived as follows:

$$\tan\sqrt{cP/(1-c_sP)} = \sqrt{cP(1-c_sP)} \tag{7}$$

by which dimensionless critical load can also be determined very easily. If let $c_s = 0$ and $c = 1$ then Eq.(7) reduces to the eigenvalue equation for determining critical load of homogenous Euler-Bernoulli beam with C-S ends.

CONCLUSIONS

A new approach for predicting the critical loads of axially compressive FGM Timoshenko beams were presented without solving the coupled differential equations with boundary conditions. For S-S, C-C and C-F (or F-C) end supports, dimensionless critical loads of FGM Timoshenko beams can be expressed in terms of those homogenous Euler-Bernoulli beams by Eq.(6). For C-S (or S-C) end constraints, the critical loads of FGM Timoshenko beams can also be determined by finding the minimum root of Eq.(7) for load parameter P as easily as to find that of Euler-Bernoulli beams.

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