# ON CONSTRUCTING ANALYTICAL SOLUTIONS FOR THE POST-BIFURCATION STATES OF A COMPRESSIBLE HYPERELASTIC TUBE UNDER COMPRESSION

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*Summary* Based on the field equations, by a method of coupled series-asymptotic expansions together with the multiple scales method, we manage to construct the explicit asymptotic solutions for the axisymmetrical post-bifurcation states of a compressible hyperelastic tube under compression. Explicit formulas for the amplitudes of post-bifurcation profiles are obtained, which provide a clear picture how the geometric parameters, the mode number and the load influence the post-bifurcation states. An interesting bifurcation phenomenon, called the octopus bifurcation, is also found. The singularities theory is used to show that for some geometrical parameters the problem is imperfection sensitive. Numerical solutions obtained by AUTO are compared with the analytical solutions, and good agreements are found.

## INTRODUCTION AND FIELD EQUATIONS

Since the field equations for hyperelastic materials comprise a formidable system of nonlinear PDEs, most studies on stabilities for such materials in literature were carried out within the framework of linearized theories. For example, Haughton and Ogden [1] and Goriely et al. [2] carried out linear stability analysis for *incompressible* hyperelastic tubes. Few analytical solutions have been obtained for the post-bifurcation states, and one exception is the work by Triantafyllidis et al. [3]. They considered the post-buckling states of a hyperelastic rectangular block under compression/tension and obtained the post-buckling asymptotic solutions by using the LSK method. Here, we adopt a method of coupled series-asymptotic expansions developed earlier by us to construct analytical (asymptotic) solutions for the post-bifurcation solutions for a compressible hyperelastic tube under compression.

We consider the axisymmetric deformations induced by a static axial force at one end with zero axial displacement at the other end. The inner and outer surfaces of the tube are traction-free. In the stress-free configuration, the inner and outer radii of the tube are respectively $a$ and $b$ and the length is $l$. The displacements of a material point can be represented as $U(R, Z) = r(R, Z) - R$, $W(R, Z) = z(R, Z) - Z$. For an isotropic compressible hyperelastic material, the strain-energy function $\Phi$ is a function of the three principle stretches of the Lagrangian strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$. The conjugate stress tensor is given by $\mathbf{T} = \frac{\partial \Phi}{\partial \mathbf{E}}$. If the strains are small, it is possible to expand $T_{ij}$ in term of the strains up to any order. The formula containing terms up to the third order material nonlinearity is

$$T_{ij} = A^{(1)}_{ijkl}E_{lk} + \frac{1}{2}A^{(2)}_{ijklmn}E_{lk}E_{nm} + \frac{1}{6}A^{(3)}_{ijklmnpq}E_{lk}E_{nm}E_{qp} + O(|E_{st}|^4), \tag{1}$$

where the coefficients can be computed once $\Phi$ is specified. The nominal stress tensor $\mathbf{S}$ is given by $\mathbf{S} = \mathbf{T}\mathbf{F}^T$.

For a static equilibrium state of the tube, we have

$$\frac{\partial S_{Rr}}{\partial R} + \frac{\partial S_{Zr}}{\partial Z} + \frac{S_{Rr} - S_{\Theta\theta}}{R} = 0, \quad \frac{\partial S_{Rz}}{\partial R} + \frac{\partial S_{Zz}}{\partial Z} + \frac{S_{Rz}}{R} = 0. \tag{2}$$

The traction-free boundary conditions are

$$S_{Rr} = 0, \;\; S_{Rz} = 0 \quad \text{at} \quad R = a, b. \tag{3}$$

The above two equations together with the traction-free boundary conditions comprise a formidable system for two unknowns $U$ and $W$. Now, we shall conduct an asymptotic analysis by exploring the small quantities involved.

## ASYMPTOTIC MODEL EQUATIONS AND SERIES-ASYMPTOTIC REDUCTIONS

We introduce the transformation $U = Ru$, $s = R^2$ and the the scalings: $s = l^2\tilde{s}$, $Z = l\tilde{z}$, $u = \frac{h}{l}\tilde{u}$, $W = h\tilde{w}$, where $h$ is the maximum distance that one tube end can move (and we make the constraint $h/l =: \varepsilon$ being small). Then, we obtain a dimensionless system for the two unknowns $\tilde{u}$ and $\tilde{w}$, which are functions of the spatial variable $z$, the variable $s$ and the small parameter $\varepsilon$ and the two geometric parameters $\nu_0 = a^2/l^2$ and $\nu_1 = b^2/l^2$. We also consider the case that $\delta = \nu_1^2 - \nu_0^2 = (a + b)(a - b)/l^2$ is small such that $O(\delta^2)$- terms can be omitted. Then, $0 \le s \le \delta$ varies in a small interval. This latter fact permits us to use Taylor series around $s = 0$ for the two unknowns. Then, from the dimensionless system we can obtain an asymptotically-valid system of eight nonlinear ODEs for eight coefficients in the Taylor series (with the variable $s$ being eliminated). Further by asymptotic expansions in $\varepsilon$, we can reduce them into two equations for

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two unknowns. For a Blatz-Ko material, one of the two equations takes the following form (due to space limit the other equation is not given here):

$$\frac{5}{4}\gamma - \frac{1}{3}U + \frac{1}{18}U^2 - \frac{2}{81}U^3 + \left(-\frac{4}{3} + \frac{1}{9}U - \frac{1}{54}U^2\right)G + \left(\frac{26}{9} - \frac{2}{27}U\right)G^2$$
$$-\frac{398}{81}G^3 - \frac{2\nu_0}{3}U_Z^2 + \delta\Bigg( -\frac{1}{24}U_Z^2 + \left(\frac{1}{3} - \frac{1}{9}U\right)U_{ZZ} - \frac{16}{9}U_{ZZ}G - \frac{2}{3}U_Z G_Z$$
$$+\frac{1}{\nu_0}\left(\frac{2}{9}U^2 - \frac{38}{81}U^3 + \left(\frac{1}{9}U - \frac{19}{54}U^2\right)G + \left(\frac{1}{72} - \frac{17}{108}U\right)G^2 - \frac{2}{81}G^3\right)\Bigg) = 0. \quad (4)$$

where $\gamma$ is the engineering stress, $aU$ and $G$ are respectively the leading terms of the radial displacement and axial strain of a material point on the inner surface.

## ANALYTICAL POST-BIFURCATION SOLUTIONS

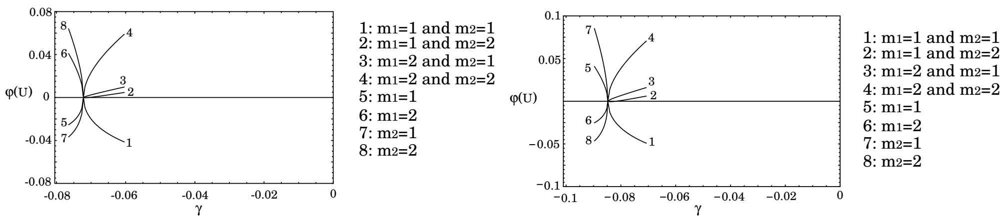
When the two ends of the tube are lubricated, asymptotically the boundary conditions for the above two equations are $U_Z = 0, \quad G_Z = 0 \quad \text{at} \quad Z = 0, l$. We use the multiple scales methods to construct the nontrival analytical solutions of these boundary-value problem (which also yield the bifurcation condition automatically). In one case, the expression for one unknown is

$$U \approx U_h + \zeta U^{(0)} + \zeta^2 U^{(1)} = U_h + 2(-1)^{m_1}(r_1\Delta\gamma)^{1/2}\cos(n_1\pi Z) + \Big(\alpha_0 + \big(2\alpha_1\cos(2n_1\pi Z) + \alpha_5\big)r_1\Big)\Delta\gamma, \quad (5)$$

where $\Delta\gamma$ is the incremental stress upon a critical load, and the coefficients are related to the geometrical parameters and the mode number $n_1$. The deflection amplitude of the inner surface is determined by $2a(r_1\Delta\gamma)^{1/2}$, which provides the explicit relation how it depends on the incremental load, the geometrical parameters and the mode number. We also perform the numerical simulations by AUTO, which confirm the analytical solutions are valid.

## IMPERFECTION SENSITIVITY

For certain geometrical parameters, there are two modes corresponding to one critical load. In this case, there are eight nontrivial branches coming out the bifurcation point. We call such a bifurcation an octopus bifurcation (see Figure 1).




(a) $\nu_0 = 0.1$ and $\rho = 0.103995$ (b) $\nu_0 = 0.22$ and $\rho = 0.115096$

**Figure 1.** Octopus bifurcations (where $\rho = \delta/\nu_0$)

We also find that in general (including both a single mode case and the two-mode case) the amplitudes $R_1$ (for mode 1) and $R_2$ (for mode 2) satisfy

$$s_0\hat{\gamma}R_1 + s_1R_1^3 + s_2R_1R_2^2 = 0, \quad t_0\hat{\gamma}R_2 + t_1R_2^3 + t_2R_2R_1^2 = 0. \quad (6)$$

By the singularities theory (see Golubitsky and Schaeffer [4]), we are able to find the universal unfolding of the normal form equations, which take the form

$$\varepsilon_1R_1^3 + pR_1R_2^2 + \varepsilon_2(-\hat{\gamma})R_1 = 0, \quad qR_2R_1^2 + \varepsilon_3R_2^3 + \varepsilon_4(-\hat{\gamma} - \sigma)R_2 = 0, \quad (7)$$

where $\sigma$ is a small constant denoting the imperfection. Then, it can be shown that for the case of two modes corresponding to the same critical load the problem is imperfection sensitive.

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