A new constitutive formulation in finite thermo-viscoelasticity

Zhuping Huang [*a)] & Guansuo Dui [**]
[*] *Department of Mechanics and Engineering Science, College of Engineering, Peking University, Beijing, 100871, China*
[**] *Institute of Mechanics, Beijing Jiaotong University, Beijing, 100044, China*

Summary Five requirements for constructing viscoelastic constitutive relations are introduced. Based on these requirements, a thermo-viscoelastic constitutive formulation is presented. First, the strain-energy function of incompressible rubber elastomers based on a molecular network model is extended to consider the material compressibility and thermal effect. The novelty of this extension is that the material parameters (or material functions) to be fitted with the experimental data have clear physical meaning and are as few as possible. Second, in order to describe the material viscosity, a set of internal variables is introduced into the above derived free energy expression of the thermo-elastic material. The new finding in this paper is that a relationship between the present internal variable theory and the constitutive theory based on the multiplicative decomposition of the deformation gradient in existing literature is established and explained, which makes the physical meaning of the internal variables more clear.

INTRODUCTION

It is known that linear viscoelastic constitutive relations may be constructed by means of the Boltzmann superposition principle. One of the well acceptable linear constitutive relations is formulated based on the concept of internal (hidden) variables (e.g., [1]). In order to extend these linear relations to the case for fully coupled nonlinear thermo-viscoelastic materials at finite deformation, the following five requirements have to be satisfied:1) it must be compatible with the second law of thermodynamics; 2) the physical meaning of these constitutive relations can be interpreted by micro- or meso-scopic deformation mechanisms under reasonable physical assumptions; 3) it should be consistent with classical theory of infinitesimal viscoelasticity; 4) material parameters (or material function) determined by experiments are as few as possible and have clear physical meaning; 5) these viscoelastic constitutive relations can reduce to those in finite thermo-elasticity during sufficiently slow (or fast) deformation process. Based on the above requirements, a new thermo-viscoelastic constitutive formulation is presented. First, the molecular network model for rubber elastomers is extended to take into account the material compressibility and thermal effect. Then, in order to describe the material viscosity, the effective stretch of molecular chains (as was given in [2]), and hence a set of internal variables are introduced into the above derived free energy expression of the thermo-elastic material. Finally, it is the first time that a new relationship between the present internal-variable theory and the constitutive theory based on the multiplicative decomposition of the deformation gradient (e.g., [3]) is derived, indicating the internal variables can also be introduced from another approach, and hence further confirming the validity of the present internal variable theory.

An extension of the molecular network model of incompressible rubber elastomers

Now the strain-energy function of incompressible rubber elastomers based on a molecular network model will be extended to take into account the material compressibility and thermal effect. This strain-energy function at the reference temperature θ_0 can be written in the form as

$$W^0 = \int_0^{\pi} \mathrm{d}\varphi \int_0^{2\pi} h(\varphi,\omega) w^0(\theta_0,\lambda) \sin\varphi \mathrm{d}\omega \tag{1}$$

where λ is the stretch of an individual chain directed along a unit vector in the spherical coordinate (r,φ,ω), $w^0(\theta_0,\lambda)$ is the strain energy of a single chain, $h(\varphi,\omega)$ is the initial orientation distribution function of molecular chains, satisfying $\int_0^{\pi} \mathrm{d}\varphi \int_0^{2\pi} h(\varphi,\omega)\sin\varphi \mathrm{d}\omega = 1$.

In particular, in the Gauss statistical theory, $w^0(\theta_0,\lambda)$ is equal to $3\mu^0(\lambda^2-1)/2$, and eq.(1) reduces to the strain-energy function of the neo-Hookean material as follows

$$W^0 = \mu^0 \frac{1}{2}(I_1 - 3) \tag{2}$$

where I_1 is the first invariant of the right Cauchy-Green tensor, μ^0 is the ground-state shear modulus.

In the case of large deformation, the non-Gauss statistics such as the eight-chain model or the full-network model may be utilized, in which the inverse Langevin function may be employed.

It is noted that, in reality, rubbers are only nearly incompressible. In order to take into account the material compressibility, the above strain-energy function for incompressible rubber elastomers should be modified. For instance, eq. (2) can be expressed as

$$W = \mu^0 \left[\frac{1}{2}(I_1 - 3) - \ln J - h(J) \right] + \lambda^0 G(J) \tag{3}$$

[a)] Corresponding author. Email: huangzp@pku.edu.cn.

where $J=\det \boldsymbol{U}$, $\boldsymbol{U}$ is the right stretch tensor, μ^0 and λ^0 are ground-state Lamé constants. In order to meet the third requirement in the introduction, the following conditions should be satisfied: $h(1)=h'(1)=0$, $h''(1)=0$ and $G(1)=G'(1)=0$, $G''(1)=1$. Obviously, there are infinitely many functions of $h(J)$ and $G(J)$ satisfying the above conditions. The novelty in this paper is the material function $G(J)$ can be uniquely determined, and for eq.(3) we have

$$G(J)=\frac{9}{4}\left[J^{2/3}-1-\frac{2}{3}(\ln J+h(J))\right] \tag{4}$$

Thus except the material function $h(J)$ describing the pressure-volume relation, no fitting parameters are required in this expression.

Equation (3) can be further extended to take into account the thermal effect. If the specific heat C_E is assumed to be a constant, the Helmholtz free energy can be written as [4]

$$\psi\ (\theta,\boldsymbol{E})=\varepsilon_0-\theta_0\eta_0+(C_E-\eta_0)(\theta-\theta_0)-C_E\theta\ln\frac{\theta}{\theta_0}+\frac{1}{\rho_0}\tilde{W} \tag{5}$$

where ρ_0, ε_0 and η_0 are the mass density, internal energy and entropy at the reference temperature θ_0, respectively. In eq. (5), $\tilde{W}$ is a linear function of the temperature, and for the neo-Hookean material, it can be given by

$$\tilde{W}=\mu^0\left(\frac{\theta}{\theta_0}\right)\left[\frac{1}{2}(I_1-3)-\ln J-h(J)\right]+\frac{9}{4}\lambda^0\left[(J^{2/3}-1)-\frac{2}{3}(\ln J+h(J))\right]-\frac{9}{2}k^0\alpha_0\left[(J^{2/3}-1)-Jh'(J)\right](\theta-\theta_0) \tag{6}$$

Again, there is no additional material parameter to be fitted with the experiments.

A new interpretation of the internal-variable theory

Now we will consider the viscous effect of the material. Suppose there are *M* kinds of molecular network with different relaxation times, and the viscous dissipation property of *m*-th (*m*=1,2,..*M*) kind of network can be modelled by a second-rank symmetric tensor $\boldsymbol{\xi}_m$ named the *m*-th internal variable. It is shown that the internal variables can only be included in the first term of the right-hand side of eq. (6), hence for the *m*-th kind network the right stretch tensor $\boldsymbol{U}$ should be replaced by the effective stretch tensor $\bar{\boldsymbol{U}}_m=\boldsymbol{U}-\boldsymbol{\xi}_m$. On the other hand, for the *m*-th kind network, the mechanical response will be purely elastic under rapidly unloading process with deformation gradient $\left(\boldsymbol{F}_e{}^m\right)^{-1}$, so the multiplicative decomposition of the deformation gradient of the *m*-th kind network can be written as $\boldsymbol{F}^m=\boldsymbol{F}_e{}^m\cdot\boldsymbol{F}_i^m$ with the polar decomposition of $\boldsymbol{F}_i^m$ being $\boldsymbol{F}_i^m=\boldsymbol{R}_e{}^m\cdot\boldsymbol{\Phi}^m$, where $\boldsymbol{\Phi}^m$ is a symmetric, positive-definite tensor. Moreover, it can be proved that $\boldsymbol{U}$ and $\boldsymbol{\Phi}^m$ have the same Lagrangian principal direction $\boldsymbol{L}_\alpha\ (\alpha=1,2,3)$, therefore, noting that the spectrum representations of $\boldsymbol{U}$ and $\boldsymbol{\Phi}^m$ are given by $\boldsymbol{U}=\sum_{\alpha=1}^{3}\lambda_\alpha\boldsymbol{L}_\alpha\otimes\boldsymbol{L}_\alpha$ and $\boldsymbol{\Phi}^m=\sum_{\alpha=1}^{3}\Phi_\alpha{}^m\boldsymbol{L}_\alpha\otimes\boldsymbol{L}_\alpha$, respectively, we can obtain $\bar{\boldsymbol{U}}_m=\sum_{\alpha=1}^{3}\left(\frac{\lambda_\alpha}{\Phi_\alpha{}^m}\right)\boldsymbol{L}_\alpha\otimes\boldsymbol{L}_\alpha$. Thus the relationship between the internal variable theory and that based on the multiplicative decomposition of the deformation gradient is established, and this relationship has not been seen in the existing literature.

CONCLUSIONS

Based on five requirements for constructing the viscoelastic constitutive relations, a new constitutive formulation in finite thermo-viscoelasticity is presented. The Gauss statistical model is used as an example to show the effectiveness of the present formulation. The internal-variable theory in thermo-viscoelasticity at finite deformation proposed in reference [2] is further developed. The new finding in this paper is that it is the first time to establish a relationship between the constitutive theory based on internal-variables and that based on the multiplicative decomposition of the deformation gradient. This makes the physical meaning of the internal variables more clear.

References

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