ANALYTICITY AND CAUSALITY OF RHEOLOGICAL MODELS

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Summary In this paper, the basic transfer functions and time-response functions of linear phenomenological models are revisited. The relation between the analyticity of a transfer function and the causality of the corresponding time-response function is extended for the case of generalized transfer functions. By using the properties of the Hilbert transform and the associated Kramers-Kroning relations it is shown that transfer functions that have a singularity $\omega = 0$ in their imaginary part should be corrected by adding a delta function in their real part. This operation ensures that the resulting time-response function is causal and is consistent with the theory of generalized functions. Accordingly, the transfer functions of classical visoelastic models presented in standard vibration handbooks are revised.

INTRODUCTION

Traditionally, the linear theory of viscoelasticity has evolved in an inductive manner starting from the "elastic spring" (Hookean Solid) and the "viscous dashpot" (Newtonian fluid) and proceeding to more comprehensive phenomenological models by linear combinations of the two aforementioned basic elements. The behavior of several isotropic materials when stressed at small deformations gradients can be satisfactorily described with combinations of "elastic springs" and "viscous dashpots" and it can be described by linear differential equations with constant coefficients of the form,

$$[\sum_{m=0}^{M} \alpha_m \frac{d^m}{dt^m}]\tau(t) = [\sum_{n=0}^{N} b_n \frac{d^n}{dt^n}]\gamma(t)\,, \tag{1}$$

where $\tau(t)$ and $\gamma(t)$ are the time histories of the stress and the small-gradient strain; α_m and b_m are restricted to real numbers and are the parameters of the constitutive model, while the order of differential m and n is restricted to integers. In this paper we are concerned with the integral representation of equation (1). Linear-viscoelastic materials obey the so-called Boltzmann superposition principle- that the output history can be obtained as the convolution of the input history after being convolved with the corresponding basic time-response function. The basic time-response functions can be obtained either by imposing an impulse or a unit-step excitation in the constitutive model or by inverting in the time domain the corresponding frequency response functions of the constitutive models. Such techniques are well known in the disciplines of rheology (Ferry 1980; Bird et al 1987; Tschoegl 1989), structural mechanics (Harris 1988) and automatic control (Bode 1959), among others.

THE HIDDEN DIRAC DELTA FUNCTION

The need for the addition of a Dirac delta function in the real part of frequency response functions, that their imaginary part has a singularity at $\omega = 0 + i0$ was discovered by Makris (1997) and is illustrated in this section by examining the simplest phenomenological model- the linear elastic spring. For a Hookean solid equation (1) reduces to

$$\tau(t) = G\gamma(t)\,, \tag{2}$$

and from equation (3) the complex dynamic modulus is merely

$$\mathcal{G}(\omega) = G + i0\,, \tag{3}$$

while equation (5) yields that the memory function,

$$q(t) = G\delta(t-0)\,. \tag{4}$$

Now, equation (9) suggests that the dynamic viscosity of the Hookean solid (strain-rate frequency response function)

$$\eta(\omega) = \eta_1(\omega) + i\eta_2(\omega) = \frac{\mathcal{G}(\omega)}{i\omega}\,, \tag{5}$$

and according to equation (5) the dynamic viscosity (impedance) of the Hookean solid is

$$\eta(\omega) = 0 - i\frac{G}{\omega}\,. \tag{6}$$

The inverse Fourier transform, $\frac{1}{2\pi}\int_{-\infty}^{+\infty} -\frac{i}{\omega}e^{i\omega t}d\omega = \frac{1}{2}\mathrm{sgn}(t)$, is well known in the literature (Morse and Feshbach 1953) and according to equation (12), given the expression offered by (6), the relaxation function of the Hookean solid assumes the expression

$$G(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}-i\frac{G}{\omega}e^{i\omega t}d\omega=\frac{1}{2}G\,\mathrm{sgn}(t)\,. \tag{7}$$

The reader recognizes that although the complex dynamic viscosity of the Hookean spring, given by (6), is a strictly proper function, the resulting relaxation function is the erroneous non-causal signum function which maintains a finite value along the entire negative time axis. In fact, equation (7) erroneously suggests that the elastic spring will produce as much response at negative times (prior to the excitation) as the response produced following the excitation. The origin of this causality violation is that although the expression of the complex viscosity, $\eta(\omega)$, given by (6) is a strictly proper frequency response function, the real and imaginary parts of $\eta(\omega)$ given by (6) are not Hilbert pairs (Tschoegl 1989). The violation of causality shown by (7) can be cured by requiring the real and imaginary parts (6) to be Hilbert pairs, or in other words to satisfy Kramers-Kronig relations

$$\eta_1(\omega)=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\eta_2(x)}{x-\omega}dx\,,\quad \eta_2(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\eta_1(x)}{x-\omega}dx\,. \tag{8}$$

The imaginary part, $-\frac{G}{\omega}$, of $\eta(\omega)$ given by (6), has as Hilbert transform the function, $G\pi\delta(\omega-0)$ - a function that is everywhere zero; yet singular at the static limit. This can be shown immediately from the second relation of (8) after making the change of variables $\xi=x-\omega$, $d\xi=dx$.

$$\eta_2(\omega)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\pi\delta(x-0)}{x-\omega}dx=\int_{-\infty}^{+\infty}\frac{\pi\delta[\xi-(-\omega+0)]}{\xi}d\xi=\frac{1}{-\omega+0}=\frac{1}{\omega} \tag{9}$$

Accordingly, the real part of the complex viscosity of the Hookean spring is $\eta_1(\omega)=\pi G\delta(\omega-0)$ (not zero) and the correct value of the complex viscosity of the Hookean spring is not equation (6) but the equation below:

$$\eta(\omega)=\pi G[\delta(\omega-0)-i\frac{1}{\pi\omega}]. \tag{10}$$

By inverting back in the time domain equation (10), the correct expression for the relaxation modulus of the Hookean spring is recovered

$$G(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\pi G[\delta(\omega-0)-i\frac{1}{\pi\omega}]e^{i\omega t}d\omega=\frac{1}{2}G[1+\mathrm{sgn}(t)]=GH(t) \tag{11}$$

where $H(t)$ is the Heaviside step function and the correct causal result is recovered (Giesekus 1995). The above calculation shows that the requirement that the real and imaginary part of the complex dynamic viscosity to be Hilbert pairs imposes the presence of a Dirac delta function in its real part which when transformed in the time domain offers the so much needed unity which lifts the signum function by the necessary amount to convert it to the causal Heaviside function. The presence of a Dirac delta function as the real part of the complex viscosity extends the concept of analyticity to generalized functions and essentially makes the reciprocal function $\frac{1}{\omega}$ well defined in the neighborhood ω=0. The intimate relation between the reciprocal function and the delta function appearing in the right hand side of (10) was first noticed by Dirac (1958).

The final result of equation (11), $G(t)=GH(t)$, has been presented in the paper by Giesekus (1995) who followed faithfully the definition of the relaxation modulus and imposed in the constitutive equation of the Hookean solid, $\tau(t)=G\gamma(t)$ a step strain excitation $\gamma(t)=\gamma H(t)$. In this paper the same result is recovered with a rigorous mathematical formulation which emerges from the fundamental relation between the causality of the time response function and the analyticity of the corresponding frequency response functions which herein is extended in the case of generalized functions. This mathematical formulation is further applied in this paper in an effort to compute the basic frequency response functions of the three-parameter solid and fluid models.

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