CONSTRAINTS AND ENERGY PROJECTION METHOD FOR DYNAMIC EQUATIONS OF MULTIBODY SYSTEMS

Jieyu Ding[*a)], Zhenkuan Pan[*]
[*]*College of Information Engineering, Qingdao University, Qingdao, 266071, China*

Summary The numerical integration method used constraints projection and energy projection is presented for differential-algebraic equations (DAEs) of multibody systems. In each step of integration which used time-step methods, the results of generalized displacements, velocities and accelerations are projected on the manifolds of the constraints and their first and second order derivatives with regard to time. And to restrain the dissipation of total energy, the energy projection is also used to build the union equations as well as the constraints projection. Numerical results show that the method presented can improve the stability of integrators and preserve the constraints, velocity constraints and acceleration constraints well, and control the changes of total energy in a small scale during long time simulation.

TIME-STEP INTEGRATION

Typical equations of motion for multibody systems under constraints are the following differential-algebraic equations

$$\begin{cases} \boldsymbol{M}(\boldsymbol{q},t)\ddot{\boldsymbol{q}} + \boldsymbol{\Phi}_{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{\lambda} = \boldsymbol{F}(\dot{\boldsymbol{q}},\boldsymbol{q},t) \\ \boldsymbol{\Phi}(\boldsymbol{q},t) = \boldsymbol{0} \end{cases} \tag{1}$$

where $\boldsymbol{q} = \boldsymbol{q}(t) \in R^n$ is a vector of generalized coordinates, $\boldsymbol{M} \in R^{n\times n}$ is the generalized mass matrix, $\boldsymbol{F} \in R^n$ is the applied forces vector, $\boldsymbol{\Phi} \in R^m$ is the holonomic constraint function, and $\boldsymbol{\lambda} \in R^m$ is Lagrange multiplier. Equation (1) is index 3 DAEs, which is more difficult to be solved. Since Newmark method and HHT method are introduced into dynamic equations of holonomic constraint systems by Cardona and Geradin[1], time-step integration methods for multibody systems are studied by many researchers, such as HHT-I3, HHT-ADD, HHT-SI2[2], Generalized-α[3], α-RATTLE[3] et al. These methods can improve the stability of integrators, but only one or two of the three kind of constraints, which are displacement constraints, velocity constraints and acceleration constraints, are preserved in higher accuracy, and others violate. Because of the damping introduced in these methods, dissipation appears in the total energy of the system. The aim of this paper is to keep all the three kind of constraints at the same time and control the changes of the total energy in small scale through the projection method.

CONSTRAINTS AND ENERGY PROJECTION

In each step of integration, the results of generalized displacements, velocities and accelerations can be projected on their constraint manifold to make sure that the new results can satisfied all the three kind of constraints at the same time. The similar projection process can be chosen for the total energy to make it not change as far as possible. Combine all these projection formulas and discrete form dynamic equations, the constraints and energy projection formulas are as following

$$\begin{cases} \boldsymbol{M}\left(\tilde{\boldsymbol{q}}_{i+1},t_{i+1}\right)\tilde{\ddot{\boldsymbol{q}}}_{i+1} + \boldsymbol{\Phi}_{\boldsymbol{q}}^{\mathrm{T}}\left(\tilde{\boldsymbol{q}}_{i+1},t_{i+1}\right)\boldsymbol{\lambda}_{i+1} - \boldsymbol{F}(\tilde{\dot{\boldsymbol{q}}}_{i+1},\tilde{\boldsymbol{q}}_{i+1},t_{i+1}) = \boldsymbol{0} \\ \boldsymbol{\Phi}(\tilde{\boldsymbol{q}}_{i+1},t_{i+1}) = \boldsymbol{0} \\ \boldsymbol{q}_{i+1} - \tilde{\boldsymbol{q}}_{i+1} + \boldsymbol{\Phi}_{\boldsymbol{q}}^{\mathrm{T}}\left(\boldsymbol{q}_{i+1},t_{i+1}\right)\boldsymbol{\mu}_{i+1} + H_{\boldsymbol{q}}^{\mathrm{T}}(\dot{\boldsymbol{q}}_{i+1},\boldsymbol{q}_{i+1},t_{i+1})\eta_{i+1} = \boldsymbol{0} \\ \boldsymbol{\Phi}(\boldsymbol{q}_{i+1},t_{i+1}) = \boldsymbol{0} \\ \dot{\boldsymbol{q}}_{i+1} - \tilde{\dot{\boldsymbol{q}}}_{i+1} + \boldsymbol{\Phi}_{\boldsymbol{q}}^{\mathrm{T}}\left(\boldsymbol{q}_{i+1},t_{i+1}\right)\boldsymbol{\nu}_{i+1} + H_{\dot{\boldsymbol{q}}}^{\mathrm{T}}(\dot{\boldsymbol{q}}_{i+1},\boldsymbol{q}_{i+1},t_{i+1})\eta_{i+1} = \boldsymbol{0} \\ \boldsymbol{\Phi}_{\boldsymbol{q}_{i+1}}\dot{\boldsymbol{q}}_{i+1} + \boldsymbol{\Phi}_{t_{i+1}} = \boldsymbol{0} \\ \ddot{\boldsymbol{q}}_{i+1} - \tilde{\ddot{\boldsymbol{q}}}_{i+1} + \boldsymbol{\Phi}_{\boldsymbol{q}}^{\mathrm{T}}\left(\boldsymbol{q}_{i+1},t_{i+1}\right)\boldsymbol{\omega}_{i+1} = \boldsymbol{0} \\ \boldsymbol{\Phi}_{\boldsymbol{q}_{i+1}}\ddot{\boldsymbol{q}}_{i+1} + \left(\boldsymbol{\Phi}_{\boldsymbol{q}_{i+1}}\dot{\boldsymbol{q}}_{i+1}\right)_{\boldsymbol{q}_{i+1}} + 2\boldsymbol{\Phi}_{\boldsymbol{q}_{i+1}t_{i+1}}\dot{\boldsymbol{q}}_{i+1} + \boldsymbol{\Phi}_{t_{i+1}t_{i+1}} = \boldsymbol{0} \\ H(\dot{\boldsymbol{q}}_{i+1},\boldsymbol{q}_{i+1},t_{i+1}) - H(\dot{\boldsymbol{q}}_{i},\boldsymbol{q}_{i},t_{i}) = 0 \end{cases} \tag{2}$$

$$\begin{aligned} \tilde{\dot{\boldsymbol{q}}}_{i+1} &= \tilde{\dot{\boldsymbol{q}}}_i + h\left[\left(1-\gamma\right)\tilde{\ddot{\boldsymbol{q}}}_i + \gamma\tilde{\ddot{\boldsymbol{q}}}_{i+1}\right] \\ \tilde{\boldsymbol{q}}_{i+1} &= \tilde{\boldsymbol{q}}_i + h\tilde{\dot{\boldsymbol{q}}}_i + \frac{h^2}{2}\left[\left(1-2\beta\right)\tilde{\ddot{\boldsymbol{q}}}_i + 2\beta\tilde{\ddot{\boldsymbol{q}}}_{i+1}\right] \end{aligned} \tag{3}$$

All the integration method such as Newmark method, HHT, Generalized-α et al., can be used in the formulas above, which can drive different constraint and energy projection method.

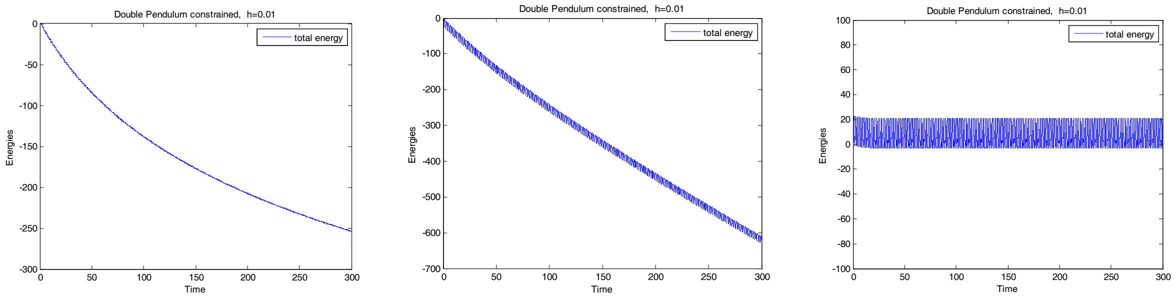
Supported by the National Natural Science Foundation of China (Grant No. 11002075, 10972110).
[a)] Corresponding author. Email: djy@qdu.edu.cn.

NUMERICAL EXAMPLE

A double pendulum system is used to verify the constraints and energy projection method. Table 1 lists the results of Runge-Kutta method, Generalized-α method, and Generalized-α constraints and energy projection method. It shows that in the long time simulation, constraints and energy projection method preserved the displacement constraints, velocity constraints and acceleration constraints very well, and the change range of total energy is small, for which the relative rate of changes to the kinetic and potential energy is controlled in 0.23%. That is also illustrated in figure 1. The cost is the increase of run time.

Table 1. Results of different methods on h=0.01, t_N=300s.

Method	Run time(s)	$\max\lvert\varepsilon(H)\rvert$	$\max\lvert\varepsilon(\Phi)\rvert$	$\max\lvert\varepsilon(\dot{\Phi})\rvert$	$\max\lvert\varepsilon(\ddot{\Phi})\rvert$
Runge-Kutta	34.5074	254.2538	3.3176e-004	0.0365	4.9087
Generalized-α	36.6914	628.0160	7.7716e-016	0.0170	5.3126
Generalized-α projection	122.8352	22.9432	7.2187e-013	7.8604e-013	7.3224e-011



(*a*) (*b*) (*c*)

Figure 1 The total energy of different methods on h=0.01, t_N=300s. (a) Runge-Kutta method (b) Generalized-α method (c) Generalized-α projection method.

CONCLUSIONS

With the projection of displacement constraints, velocity constraints and acceleration, as well as the total energy of the multibody system, a constraints and energy projection method is presented. It can be used in long time simulation for its good properties on the preservation of all three kind of constraints and the control of the changes range of total energy.

References

[1] Cardona A., Geradin M.: Time Integration of the Equations of Motion in Mechanism Analysis. *Comput. & Structures*, **33**: 801-820, 1989.

[2] Negrut D., Jay L. O., Khude N.: A Discussion of Low-order Numerical Integration Formulas for Rigid and Flexible Multibody Dynamics, *Journal of Computational and Nonlinear Dynamics*, **4**:1-11, 2009.

[3] Yao T. Q., Chi Y. L., Huang Y. Y.: New Generalized-α Algorithms for Multibody Dynamics, *Journal of Mechanical Engineering*, **45**(**10**): 53-60, 2009.

[4] Ma X. T., Chen L. P., Zhang Y. Q.: RATTLE Method for Dissipative Constrained Hamiltonian Systems. In: C. Xiong et al. (Eds.): ICIRA 2008, Part I, LNAI 5314:294-303, 2008.