THE STATISTICAL SECOND-ORDER TWO-SCALE ANALYSIS FOR THERMO-ELASTIC COUPLED PERFORMANCE OF THE COMPOSITE STRUCTURE WITH CONSISTENT RANDOM DISTRIBUTION OF GRAINS

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Summary: The consistent random distribution denotes that the probability model is same everywhere in composites. The statistical second-order two-scale formulation and its algorithm procedure for displacement and temperature of its coupled thermo-elastic problem are given, and the numerical simulation is performed.

MODELING AND FORMULATION

The fully coupled thermo-elastic problem for the structure of consistent random distribution composites is expressed as follows

$$\begin{cases}
\rho^\varepsilon(x,\omega)c^\varepsilon(x,\omega)\dfrac{\partial T^\varepsilon(x,\omega,t)}{\partial t}+\tilde{T}\beta_{ij}^\varepsilon(x,\omega)\dfrac{\partial}{\partial t}\left(\dfrac{\partial u_i^\varepsilon(x,\omega,t)}{\partial x_j}\right)-\dfrac{\partial}{\partial x_i}\left(k_{ij}^\varepsilon(x,\omega)\dfrac{\partial T^\varepsilon(x,\omega,t)}{\partial x_j}\right)=h(x,t)\ in\,\Omega\times(0,t_{end})\\
\rho^\varepsilon(x,\omega)\dfrac{\partial^2 u_i^\varepsilon(x,\omega,t)}{\partial t^2}-\dfrac{\partial}{\partial x_j}\left(C_{ijkl}^\varepsilon(x,\omega)\dfrac{\partial u_k^\varepsilon(x,\omega,t)}{\partial x_l}-\beta_{ij}^\varepsilon(x,\omega)\left(T^\varepsilon(x,\omega,t)-\tilde{T}\right)\right)=f_i(x,t)\quad in\,\Omega\times(0,t_{end})\\
\mathbf{u}^\varepsilon(x,\omega,t)=\hat{u}(x,t)\qquad on\ \Gamma_u\times(0,t_{end})\\
T^\varepsilon(x,\omega,t)=\hat{T}\left(x,t\right)\qquad on\ \Gamma_T\times(0,t_{end})\\
n_i k_{ij}^\varepsilon(x,\omega)\dfrac{\partial T^\varepsilon}{\partial x_j}=\bar{q}(x,t),\quad on\ \Gamma_q\times(0,t_{end})\\
n_i\left(C_{ijkl}^\varepsilon(x,\omega)\dfrac{\partial u_k^\varepsilon(x,\omega,t)}{\partial x_l}-\beta_{ij}^\varepsilon(x,\omega)\left(T^\varepsilon(x,\omega,t)-\tilde{T}\right)\right)=\sigma_j(x,t)\quad on\ \Gamma_t\times(0,t_{end})\\
\mathbf{u}^\varepsilon(x,\omega,0)=0,\quad \dfrac{\partial\mathbf{u}^\varepsilon}{\partial t}(x,\omega,0)=0,\quad T^\varepsilon(x,\omega,0)=\tilde{T}\left(x\right),\ in\ \Omega
\end{cases}\tag{1}$$

In global domain Ω , $\omega=\left\{\omega^s, x\in\varepsilon Q^s\subset\Omega\right\}$ is the set of samples[1,2] and Q^s is the cell corresponding with ω^s .

By constructive way the temperature and displacement fields can be expanded into a series in the following form

$$T^\varepsilon(x,\omega^s,t)=T_0(x,t)+\varepsilon M_{\alpha_1}(y,\omega^s)\frac{\partial T_0(x,t)}{\partial x_{\alpha_1}}+\varepsilon^2\left(M_{\alpha_1\alpha_2}(y,\omega^s)\frac{\partial^2 T_0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}+R_{\alpha_1\alpha_2}(y,\omega^s)\frac{\partial^2 u_{0\alpha_1}(x,t)}{\partial t\partial x_{\alpha_2}}+Q_2(y,\omega^s)\frac{\partial T_0(x,t)}{\partial t}\right)$$

$$\mathbf{u}^\varepsilon(x,\omega^s,t)=\mathbf{u}_0(x,t)+\varepsilon\left(\mathbf{N}_{\alpha_1}(y,\omega^s)\frac{\partial\mathbf{u}_0(x,t)}{\partial x_{\alpha_1}}-\mathbf{P}_0(y,\omega^s)\left(T_0(x,t)-\tilde{T}\right)\right)+\varepsilon^2\left(\mathbf{N}_{\alpha_1\alpha_2}(y,\omega^s)\frac{\partial^2\mathbf{u}_0(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}-\mathbf{P}_{\alpha_1}(y,\omega^s)\frac{\partial T_0(x,t)}{\partial x_{\alpha_1}}-\mathbf{F}_2(y,\omega^s)\frac{\partial^2\mathbf{u}_0(x,t)}{\partial t^2}\right)$$

where $T_0(x,t)$ and $\mathbf{u}_0(x,t)$ are the homogenization solutions of the homogenization equation（2）

$$\begin{cases}
\widehat{S}\dfrac{\partial T_0}{\partial t}+\tilde{T}\widehat{\beta}_{ij}\dfrac{\partial^2 u_{0i}}{\partial t\partial x_j}-\dfrac{\partial}{\partial x_i}\left(\widehat{k}_{ij}\dfrac{\partial T_0}{\partial x_j}\right)=h\qquad in\ \Omega\times(0,t_{end})\\
\widehat{\rho}\dfrac{\partial^2 u_{0i}}{\partial t^2}-\dfrac{\partial}{\partial x_j}\left(\widehat{C}_{ijkl}\dfrac{\partial u_{0k}}{\partial x_l}-\widehat{\beta}_{ij}\left(T_0-\tilde{T}\right)\right)=f_i\quad in\ \Omega\times(0,t_{end})\\
\mathbf{u}_0(x,t)=0,\ \ on\ \Gamma_u\times(0,t_{end}),\quad T_0(x,t)=\hat{T}\left(x,t\right),\ on\ \Gamma_T\times(0,t_{end})\\
n_i\widehat{k}_{ij}\dfrac{\partial T_0}{\partial x_j}=\bar{q}(x,t),\quad on\ \Gamma_q\times(0,t_{end})\\
n_i\left(\widehat{C}_{ijkl}\dfrac{\partial u_{0k}}{\partial x_l}-\widehat{\beta}_{ij}\left(T_0-\tilde{T}\right)\right)=\sigma_j\left(x,t\right),\quad on\ \Gamma_t\times(0,t_{en})\\
\mathbf{u}_0(x,0)=0,\ \dfrac{\partial\mathbf{u}_0}{\partial t}(x,0)=0,\ T_0(x,0)=\tilde{T}\left(x\right),\ in\ \Omega
\end{cases}\tag{2}$$

$M_{\alpha_1}(y,\omega^s)$ $\mathbf{N}_{\alpha_1}(y,\omega^s)$, $\mathbf{P}_0(y,\omega^s)$ are defined by three following equations, respectively,

$$\begin{cases}
-\dfrac{\partial}{\partial y_i}\left(k_{ij}(y,\omega^s)\dfrac{\partial M_{\alpha_1}(y,\omega^s)}{\partial y_j}\right)=\dfrac{\partial}{\partial y_i}\left(k_{i\alpha_1}(y,\omega^s)\right)\quad y\in Q\\
M_{\alpha_1}(y,\omega^s)=0,\qquad y\in\partial Q
\end{cases},\quad
\begin{cases}
-\dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y,\omega^s)\dfrac{\partial\mathbf{N}_{\alpha_1 km}(y,\omega^s)}{\partial y_l}\right)=\dfrac{\partial}{\partial y_j}\left(C_{ijm\alpha_1}(y,\omega^s)\right)\quad y\in Q\\
\mathbf{N}_{\alpha_1 m}(y,\omega^s)=0\qquad y\in\partial Q
\end{cases}$$

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$$\begin{cases} -\dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y,\omega^s)\dfrac{\partial \mathbf{P}_{0k}(y,\omega^s)}{\partial y_l} \right) = \dfrac{\partial}{\partial y_j}\left(\beta_{ij}(y,\omega^s)\right) & y\in Q \\ \mathbf{P}_0(y,\omega^s)=0 & y\in\partial Q \end{cases}$$

Homogenization coefficients and expected homogenization coefficients are defined as follows

$$\hat{S}(\omega^s)=\frac{1}{|Q|}\int_Q\left(\rho(y,\omega^s)c(y,\omega^s)-\tilde{T}\beta_{ij}(y,\omega^s)\frac{\partial \mathbf{P}_{0i}(y,\omega^s)}{\partial y_j}\right)dy\,,\ \hat{\beta}_{ij}(\omega^s)=\frac{1}{|Q|}\int_Q\left(\beta_{ij}(y,\omega^s)+\beta_{kl}(y,\omega^s)\frac{\partial \mathbf{N}_{jki}(y,\omega^s)}{\partial y_l}\right)dy$$

$$\hat{k}_{ij}(\omega^s)=\frac{1}{|Q|}\int_Q\left(k_{ij}(y,\omega^s)+k_{ik}(y,\omega^s)\frac{\partial M_j(y,\omega^s)}{\partial y_k}\right)dy\,,\qquad \hat{\rho}(\omega^s)=\frac{1}{|Q|}\int_Q\rho(y,\omega^s)d\xi y$$

$$\hat{C}_{ijkl}(\omega^s)=\frac{1}{|Q|}\int_Q\left(C_{ijkl}(y,\omega^s)+C_{ijm\alpha_1}(y,\omega^s)\frac{\partial \mathbf{N}_{lmk}(\xi,\omega^s)}{\partial y_{\alpha_1}}\right)dy\,,$$

$$\hat{S}=\frac{\sum_{s=1}^{M}\hat{S}(\omega^s)}{M},M\to+\infty\quad \hat{\beta}_{ij}=\frac{\sum_{s=1}^{M}\hat{\beta}_{ij}(\omega^s)}{M},M\to+\infty\quad \hat{k}_{ij}=\frac{\sum_{s=1}^{M}\hat{k}_{ij}(\omega^s)}{M},M\to+\infty\quad \hat{\rho}=\frac{\sum_{s=1}^{M}\hat{\rho}(\omega^s)}{M},M\to+\infty\quad \hat{C}_{ijkl}=\frac{\sum_{s=1}^{M}\hat{C}_{ijkl}(\omega^s)}{M},M\to+\infty$$

Where M is the number of samples.

$M_{\alpha_1\alpha_2}(y,\omega^s)$, $R_{\alpha_1\alpha_2}(y,\omega^s)$, $Q_2(y,\omega^s)$, $\mathbf{N}_{\alpha_1\alpha_2}(y,\omega^s)$, $\mathbf{P}_{\alpha_1}(y,\omega^s)$, $\mathbf{F}_2(y,\omega^s)$ are defined by six following equations

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(k_{ij}(y,\omega^s)\dfrac{\partial M_{\alpha_1\alpha_2}(y,\omega^s)}{\partial y_j}\right)=-\hat{k}_{\alpha_1\alpha_2}+k_{\alpha_1\alpha_2}(y,\omega^s) & \\ +k_{\alpha_2 j}(y,\omega^s)\dfrac{\partial M_{\alpha_1}}{\partial y_j}+\dfrac{\partial}{\partial y_i}\left(k_{i\alpha_2}(y,\omega^s)M_{\alpha_1}(y,\omega^s)\right) & y\in Q \\ M_{\alpha_1\alpha_2}(y,\omega^s)=0 & y\in\partial Q \end{cases},$$

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(k_{ij}(y,\omega^s)\dfrac{\partial R_{\alpha_1\alpha_2}(y,\omega^s)}{\partial y_j}\right) & \\ =\tilde{T}\left(\hat{\beta}_{\alpha_1\alpha_2}-\beta_{\alpha_1\alpha_2}(y,\omega^s)-\beta_{ij}(y,\omega^s)\dfrac{\partial \mathbf{N}_{\alpha_2 i\alpha_1}(y,\omega^s)}{\partial y_j}\right) & y\in Q \\ R_{\alpha_1\alpha_2}(y,\omega^s)=0 & y\in\partial Q \end{cases}$$

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left(k_{ij}(y,\omega^s)\dfrac{\partial Q_2(y,\omega^s)}{\partial y_j}\right) & \\ =\tilde{T}\beta_{ij}(y,\omega^s)\dfrac{\partial \mathbf{P}_{0i}(y,\omega^s)}{\partial y_j}-\rho(y,\omega^s)c(y,\omega^s)+\hat{S} & y\in Q \\ Q_2(y,\omega^s)=0 & y\in\partial Q \end{cases},$$

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y,\omega^s)\dfrac{\partial \mathbf{N}_{\alpha_1\alpha_2 km}(y,\omega^s)}{\partial y_l}\right)=\hat{C}_{i\alpha_1 m\alpha_2}-C_{i\alpha_1 m\alpha_2}(y,\omega^s) & \\ -C_{i\alpha_1 kj}(y,\omega^s)\dfrac{\partial \mathbf{N}_{\alpha_2 km}(y,\omega^s)}{\partial y_j}-\dfrac{\partial}{\partial y_j}\left(C_{ijk\alpha_2}(y,\omega^s)\mathbf{N}_{\alpha_1 km}(y,\omega^s)\right) & y\in Q \\ \mathbf{N}_{\alpha_1\alpha_2 m}(y,\omega^s)=0 & y\in\partial Q \end{cases}$$

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y,\omega^s)\dfrac{\partial \mathbf{P}_{\alpha_1 k}(y,\omega^s)}{\partial y_l}\right)=\hat{\beta}_{i\alpha_1}-\beta_{i\alpha_1}(y,\omega^s)-\dfrac{\partial}{\partial y_j}\left(\beta_{ij}(y,\omega^s)M_{\alpha_1}(y,\omega^s)\right) & \\ -C_{i\alpha_1 kl}(y,\omega^s)\dfrac{\partial \mathbf{P}_{0k}(y,\omega^s)}{\partial y_l}-\dfrac{\partial}{\partial y_l}\left(C_{ilk\alpha_1}(y,\omega^s)\mathbf{P}_{0k}(y,\omega^s)\right) & y\in Q \\ \mathbf{P}_{\alpha_1 k}(y,\omega^s)=0 & y\in\partial Q \end{cases},$$

$$\begin{cases} \dfrac{\partial}{\partial y_j}\left(C_{ijkl}(y,\omega^s)\dfrac{\partial \mathbf{F}_{2km}(y,\omega^s)}{\partial y_l}\right)=\delta_{mi}\left(\hat{\rho}-\rho(y,\omega^s)\right) & y\in Q \\ \mathbf{F}_{2m}(y,\omega^s)=0 & y\in\partial Q \end{cases}$$

Numerical experiments

Time(s)	$\Vert e_{T0}\Vert_{H^1}/\Vert \mathrm{T}_\varepsilon^h\Vert_{H^1}$	$\Vert e_{T1}\Vert_{H^1}/\Vert \mathrm{T}_\varepsilon^h\Vert_{H^1}$	$\Vert e_{T2}\Vert_{H^1}/\Vert \mathrm{T}_\varepsilon^h\Vert_{H^1}$
t=1	0. 149754	0. 093325	0. 0585615
t=4	0. 189948	0. 0692227	0. 053769
t=15	0. 192166	0. 0546715	0. 0521337
t=18	0. 192234	0. 0545758	0. 0521759

Table 1 comparison with computing results of H^1 norm (Temperature)

More results will be given in full paper.

References

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