BOUNDS ON THE EFFECTIVE ELASTIC MODULI OF THREE-PHASE COMPOSITES WITH COATED SPHERICAL INCLUSIONS

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Summary A new model is put forward to bound the effective elastic moduli of three-phase composites with coated spherical inclusions. According to the principles of minimum potential energy and minimum complementary energy, the upper and lower bounds on the effective elastic moduli of three-phase composites with coated inclusions are derived. The effects of the geometrical and materials's parameters of coated spherical inclusions on the effective elastic moduli of composites are analyzed in detail.

INTRODUCTION

The problem of determining the effective elastic moduli of statistically homogeneous composites has an extensive history. Several methods have been developed to derive bounds on effective parameters of composites. Hashin and Shitrikman [1] have derived bounds for effective elastic moduli for statistically homogeneous and isotropic materials by introducing polarization fields and applying variational principles, respectively. These bounds are expressed in terms of the volume fraction, which is the simplest statistical information related to the effective properties of two-phase composites. Walpole [2] has rederived Hashin-Shtrikman bounds by using the classical extremum principles with the polarization concept in a manner indicated by Hill [3]. Other derivations and generalizations have been given by Willis [4], Kröner [5], and Milton and Kohn [6]. Just as pointed out by Hashin [7], it is necessary to incorporate additional geometrical information to improve the bounds. The purpose of the present paper is to give the bounds on the effective elastic moduli of three-phase composites with coated spherical inclusions. According to author's knowledge, the bounds on the effective properties of three-phase composites consisting of the matrix, coated layers and spherical inclusions are not found.

UPPER BOUND

Consider a statistically homogeneous and isotropic three-phase composite consisting of the matrix, coated layers and spherical inclusions. It is assumed that the interfaces between phases are perfectly bonded. The trial displacement fields in the matrix, coated layers and spherical inclusions are constructed to satisfy continuity interface conditions between phases. According to the principle of minimum potential energy, the upper bounds on the effective elastic moduli of three-phase composites with coated spherical inclusions are derived as follows

$$K^E \le c_M K^M + c_C K^C + c_I K^I - \frac{c_I\left(K^I - K^C\right)^2}{K^I - K^C + \left(K^C + \frac{4}{3}G^C\right)\frac{\lambda_K^5 - 1}{15\left(\lambda_K - 1\right)^2}} \tag{1}$$

$$G^E \le c_M G^M + c_C G^C + c_I G^I - \frac{c_I\left(G^I - G^C\right)^2}{G^I - G^C + \left(2K^C + \frac{17}{3}G^C\right)\frac{\lambda_G^5 - 1}{50\left(\lambda_G - 1\right)^2}} \tag{2}$$

where $K^E(G^E)$ is the effective bulk (shear) modulus of three-phase composites with coated spherical inclusions, $K^M(G^M)$, $K^C(G^C)$ and $K^I(G^I)$ are the bulk (shear) moduli of the matrix, coated layers and spherical inclusions, respectively, c_M, c_C and c_I represent the volume fractions of the matrix, coated layers and spherical inclusions, respectively, and λ_K and λ_G are geometrical parameters. In contrast to the upper bounds given by Hashin and Shtrikman [1], the present upper bounds are still finite when the bulk and shear moduli of spherical inclusions tend to infinity. Just as pointed out by Hashin [7], all of the improved bounds in terms of higher order statistical information such as three-point correlations do not provide a practical answer to this problem.

LOWER BOUND

For the present model, the trial stress fields in the matrix, coated layers and spherical inclusions are constructed to satisfy the continuity interface conditions of traction between phases. According to the principle of minimum complementary energy, the lower bounds on the effective elastic moduli of three-phase composites with coated spherical inclusions are derived as follows

$$K^E \geq \left[\frac{c_M}{K^M} + \frac{c_C}{K^C} + \frac{c_I}{K^I} - \frac{c_I \left(\frac{1}{K^I} - \frac{1}{K^C} \right)^2}{\frac{1}{K^I} - \frac{1}{K^C} + \left(\frac{1}{K^C} + \frac{3}{4G^C} \right) \frac{\lambda_k^3}{\lambda_k^3 - 1}} \right]^{-1} \tag{3}$$

$$G^E \geq \left[\frac{c_M}{G^M} + \frac{c_C}{G^C} + \frac{c_I}{G^I} - \frac{c_I \left(\frac{1}{G^I} - \frac{1}{G^C} \right)^2}{\frac{1}{G^I} - \frac{1}{G^C} + \frac{5}{7} \left(\frac{4}{3K^C} + \frac{19}{G^C} \right) \frac{\left(\lambda_g^7 - 1 \right) \lambda_g^3}{\psi\left(\lambda_g \right)}} \right]^{-1} \tag{4}$$

where λ_k and λ_g are geometrical parameters and $\psi(\lambda_g)$ is the function of λ_g. In contrast to the lower bounds given by Hashin and Shtrikman [1], the present lower bounds are non-zeroes when the bulk and shear moduli of spherical inclusions tend to zero.

ANALYSIS AND DISCUSSIONS

To verify the present bounds, we consider a specific case that the three-phase composite degenerates into the two-phase one and compare the present bounds with Hashin-Shtrikman bounds. Fig.1 shows the variation of the effective bulk modulus of two-phase composites with the volume fraction of spherical inclusions. Here, the upper bound for $\nu_K = 3$ and the lower bound for $\eta_k = 1$ are given, respectively. From Fig.1, it can be seen that the upper bound for $\nu_K = 3$ provides a significant improvement over Hashin-Shtrikman upper bound and the lower bound for $\eta_k = 1$ coincides with Hashin-Shtrikman lower bound. Fig.2 shows the variation of the effective shear modulus of composites with the volume fraction of spherical inclusions. Here, the upper bound for $\nu_G = 3$ and the lower bound for $\eta_g = 1$ are given, respectively. From Fig.2, it can be seen that the upper bound for $\nu_G = 3$ provides a significant improvement over Hashin-Shtrikman upper bound and the lower bound for $\eta_g = 1$ is lower than Hashin-Shtrikman lower bound.

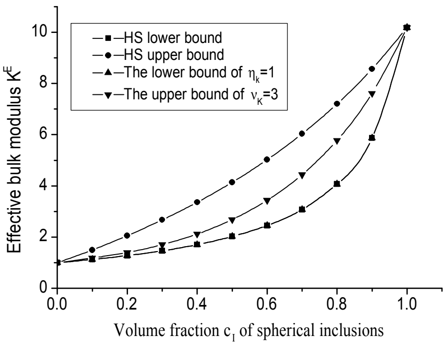

Fig.1 Comparison of the present bounds with Hashin-Shtrikman bounds for $\eta_k = 1$ and $\nu_K = 3$.

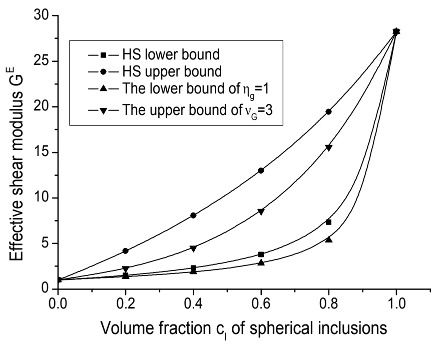

Fig.2 Comparison of the present bounds with Hashin-Shtrikman bounds for $\eta_g = 1$ and $\nu_G = 3$

CONCLUSIONS

In contrast to Hashin-Shtrikman [1] bounds and higher-order bounds, the present bounds are still finite when the elastic moduli of spherical inclusions tend to infinity and zero, respectively. It should be mentioned that the present method is simple and needs not calculate the complex integrals of multi-point correlation functions. This implies that the present bounds can be used to test various approximations on the effective elastic moduli of composites with spherical inclusions as one benchmark.

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