ON THE WAVE CATCHING-UP PHENOMENON IN A TWO-LAYER NONLINEARLY ELASTIC COMPOSITE BAR

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Summary We consider waves in a two-layer nonlinearly elastic composite bar induced by a compressive impact. Multiple reflections cause a tensile wave being transmitted into the second layer. The phenomenon that the tensile wave catches the first transmitted compressive wave is studied analytically and numerically. It is shown that there are two wave patterns for this catching-up phenomenon to happen. Besides a general mathematical theory, asymptotic solutions are also constructed, which provide the insights on the requirement of the constitutive relation, the time and place at which the catching takes place, and how the initial impact, material and geometric parameters influence the solutions. Numerical solutions are also obtained, confirming the validity of the analytical results.

INTRODUCTION AND THE FORMULATION OF THE PROBLEM

Waves in nonlinearly elastic composite materials have been extensively studied. For example, Andrianov et al [1] used an asymptotic homogenization method to study the dynamical properties of layered composite Murnaghan materials. Parnell [2] considered wave propagation in a pre-stressed nonlinearly elastic composite bar and the focus is on determining the effective incremental response. Many people have also carried out numerical computations on waves in composite materials. E.g., Berezovski et al [3] used a finite volume method to study waves in layered nonlinear heterogeneous materials due to impact. One focus of these works is how the cell structure influences the waves. The present study has a different focus: Whether and how the material nonlinearity can be explored to generate the phenomenon that a destructive tensile wave can catch a transmitted compressive wave. This will reduce the strength of the latter.

More specifically, we consider waves in a two-layer composite bar due to a compressive impact, with the first layer composed of a linearly elastic materials of finite length h (called material 1) and the second layer composed of a nonlinearly elastic material of infinite length (called material 2). In a Lagrangian description, the governing system of equations of motion reads

$$\rho v_t = \sigma_x, \quad \gamma_t = v_x, \tag{1}$$

where γ, v, σ are smooth functions and denote the strain, particle velocity and nominal stress respectively. At a moving strain discontinuity $x = s(t)$, the jump conditions are

$$\dot{s}[\gamma] + [v] = 0, \quad \dot{s}\rho[v] + [\sigma] = 0, \tag{2}$$

where $[f] = f(t, s(t) + 0) - f(t, s(t) - 0)$. The constant density ρ and the stress-strain relation are defined as

$$\rho = \begin{cases} \rho_1, & 0 \leqslant x < h \\ \rho_2, & h < x < \infty \end{cases} \qquad \sigma = \begin{cases} E_1\gamma, & 0 \leqslant x < h,\ \gamma > -1 \\ E_2 f(\gamma), & h < x < \infty,\ \gamma > -1. \end{cases} \tag{3}$$

The initial-boundary value conditions are given by

$$v(0, x) = 0,\ \gamma(0, x) = 0 \ \text{ for } \ x > 0 \ \text{ and } \ \sigma(t, 0) = \begin{cases} A, & 0 < t \leqslant T, \\ 0, & t > T. \end{cases} \tag{4}$$

We aim at investigating the catching-up phenomenon that the tensile wave (arising due to multiple reflections) catches the first transmitted compressive wave in the second layer.

MAIN RESULTS FOR THE CATCHING-UP PHENOMENON

It can be shown that if material 2 is also linear the tensile wave can arise when the impedance ratio of two materials is less than 1 but it cannot catch the compressible wave. It turns out that, when the constitutive curve $f(\gamma)$ (see (3)) of material 2 is nonlinear and convex, there are two wave patterns for the catching-up phenomenon to happen (see Figure 1). Since the self-similar structure of our problem, the strains and particle velocities are piecewise constant (cf. Dai and Kong [4]). By the jump conditions (2), it can be shown that once the three unknowns γ_1^+, γ_2^+ and γ^- are solved, all the other constant quantities can be subsequently determined. Moreover, each of them satisfies an algebraic equation containing some integral. It is not possible to get the explicit formulas for γ_1^+, γ_2^+ and γ^-, however asymptotic solutions can be constructed when the amplitude of the impact is small (in comparison with the Young modulus of material 2).

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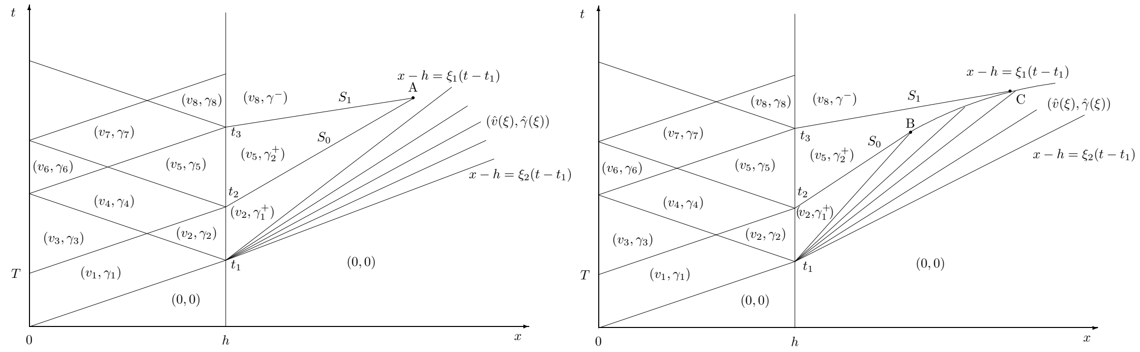

Figure 1. *The catching-up phenomenon in the composite bar. Case I: $T \geqslant T_*$ (left), the shock wave S_1 first captures the shock wave S_0; Case II: $T < T_*$ (right), the shock wave S_1 first penetrates the rarefaction wave and then is caught by the shock wave S_1, where* $T_* \doteq 2t_1 s_1(s_0 - c(\gamma_1^+))/[s_0(s_1 - c(\gamma_1^+))], c(\gamma) = (\sigma'(\gamma)/\rho_2)^{1/2}, t_1 = h/c_1, t_2 = T + t_1 = (1+\theta)t_1, t_3 = 3t_1$.

When $\varepsilon = -2A/E_2 > 0$ is small, we manage to obtain the following asymptotic results:

$$\gamma_1^+ \approx -\frac{1}{\beta+1}\varepsilon,\ \gamma_2^+ \approx -\frac{\beta\,[f''(0)]^2}{96(1+\beta)^4}\varepsilon^3,\ \gamma^- \approx \frac{1-\beta}{(1+\beta)^2}\varepsilon,\ \frac{s_0}{c_2} \approx 1 - \frac{f''(0)}{4(1+\beta)}\varepsilon,\ \frac{s_1}{c_2} \approx 1 + \frac{(1-\beta)f''(0)}{4(1+\beta)^2}\varepsilon. \tag{5}$$

Here, $c_i = (E_i/\rho_i)^{1/2} (i = 1, 2)$ are the speeds of sound waves in materials 1 and 2, $\alpha \doteq c_1/c_2, \beta \doteq \rho_1 c_1/\rho_2 c_2$. One can see that γ^- is positive when $\beta < 1$, thus the shock S_1 is tensile. Also, since $f''(0) > 0$, the tensile wave S_1 is faster than the compressive shock S_0 ($s_1 > s_0$). We point out that the shock waves in both cases actually satisfy the Lax's entropy conditions [5] since $f(\gamma)$ is convex. All the other quantities can then be determined.
First, the critical time T_* (cf. Figure 1) which separates the two wave patterns is given by $(2\beta+2)/(\beta+3)$ (to leading order). For $T \geqslant T_*$, the time and location (point A in Figure 1) at which the tensile wave S_1 catches the compressive shock S_0 are given by

$$t_A/t_1 \approx 2(\beta+1)^2(2-\theta)/[f''(0)\varepsilon],\ x_A/h \approx 2(2-\theta)(\beta+1)^2/[\alpha f''(0)\varepsilon]. \tag{6}$$

For $T < T_*$, the time and location (point B in Figure 1) at which the tensile wave S_1 catches the varying-speed compressive shock (the curve BC) are given by

$$t_C/t_1 \approx 1+X/\varepsilon,\ x_C/h \approx 1+X/(\alpha\varepsilon),\ X = 8(\beta+1)/[f''(0)(\theta+(1-\beta)/(\beta+1)+\sqrt{\theta^2+2\theta(1-\beta)/[(\beta+1)]})]. \tag{7}$$

The above analytical expressions reveal clearly the roles played by the constitutive relation, the strength of the initial impact and the material and geometrical parameters. Numerical simulations further confirm the validity of our analytical results for $\varepsilon < 0.3$.

CONCLUSIONS

For waves in a two-layer composite bar, it is found that if the impedance ratio is less than 1 a tensile wave can be transmitted into the second layer. Then, we show that if the constitutive curve of the second material is convex there are two cases in which it can catch the first transmitted compressive wave. The theory, analytical and numerical solutions presented in this paper may provide a new possible way for designing certain structures for impact protection purpose.

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