EVOLUTION OF SOLITARY WAVES IN MINDLIN TYPE MICROSTRUCTURED MATERIALS

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Summary Wave propagation in microstructured solids is studied making use of the Mindlin–Engelbrecht–Pastrone model. The main attention is paid to the influence of nonlinear and dispersive effects on the evolution of solitary waves.

INTRODUCTION AND GOVERNING EQUATIONS

The model under consideration is derived by Jüri Engelbrecht and Franco Pastrone and it describes wave propagation in nonlinear dispersive media with the Mindlin-type microstructure [1, 2, 3]. In this model the microelement is taken as a deformable cell with an additional assumption that the deformation gradient is small. The later allows one to express microdeformation in terms of macrodisplacement.

In order to derive equations of motion for the nonlinear case, the free energy is defined as follows:

$$W = \frac{A}{2}u_x^2 + \frac{B}{2}\varphi^2 + \frac{C}{2}\varphi_x^2 + D\varphi u_x + \frac{N}{6}u_x^3 + \frac{M}{6}\varphi_x^3. \tag{1}$$

Here material parameters A, B, C, D correspond to the linear part of the model and N, M to the nonlinear part, u is the macrodisplacement, φ – the microdeformation, and partial derivatives are denoted by subscripts. After introducing the Lagrangian $L = K - W$, where the kinetic energy $K = \frac{1}{2}\rho u_t^2 + \frac{1}{2}I\varphi_t^2$, one can formulate balance laws separately for the macro- and microscale making use of Euler–Lagrange equations (ρ is the macroscale density and I the microinertia). For further analysis dimensionless variables $X = \frac{x}{L_o},\ T = \frac{\sqrt{A}t}{\sqrt{\rho}L_o},\ U = \frac{u}{U_o}$, and parameters $\delta = \frac{l_o^2}{L_o^2},\ \varepsilon = \frac{U_o}{L_o}$ are introduced. Here U_o and L_o are the amplitude and the wavelength of the initial excitation, and l_o is the characteristic scale of the microstructure. Making use of these dimensionless variables equations of motion take the following form:

$$U_{TT} = \frac{DL_o}{AU_o}\varphi_X + \frac{NU_o}{AL_o}U_X U_{XX} + U_{XX}, \qquad \varphi_{TT} = \frac{C\rho}{AI}\varphi_{XX} - \frac{B\rho L_o^2}{AI}\varphi - \frac{D\rho U_o L_o}{AI}U_X + \frac{M\rho}{AIL_o}\varphi_X\varphi_{XX}. \tag{2}$$

Equations (2) are referred to as the full system of equations (the FSE for short) below. Making use of the slaving principle allows one to derive a single hierarchical equation in terms of macrodisplacement U from the FSE:

$$U_{TT} - bU_{XX} - 0.5\mu\left(U_X^2\right)_X - \delta\left(\beta U_{TT} - \gamma U_{XX} + 0.5\lambda\sqrt{\delta}U_{XX}^2\right)_{XX} = 0. \tag{3}$$

The constants in equation (3) are related to the material and geometrical parameters as follows: $b = 1 - \frac{D^2}{AB},\ \mu = \frac{NU_o}{AL_o}$, $\beta = \frac{ID^2}{\rho l_o^2 B^2},\ \gamma = \frac{CD^2}{AB^2 l_o^2},\ \lambda = \frac{D^3 MU_o}{AB^3 l_o^3 L_o}$. Equation (3) can be considered as an approximation of the FSE (2), it is hierarchical in Whitham's sense [4] and is referred to as the hierarchical equation (HE) below. For the full procedure of derivation of the FSE and HE consult papers [2, 3] and references therein.

STATEMENT OF THE PROBLEM

The main goals of the present paper are to analyse (i) the influence of the dispersive and nonlinear effects on the evolution of solitary pulses, and (ii) how do the results of linear dispersion analysis can be applied for estimation of the concurrency between solutions of the FSE (2) and the HE (3).

In the present paper we combine the linear material parameters into two new parameters

$$\gamma_A^2 = 1 - b = \frac{D^2}{AB}, \qquad \gamma_1^2 = \frac{\gamma}{\beta} = \frac{\rho C}{AI} \tag{4}$$

describing macro- and microstructure and interaction between them. Parameter γ_1 can be interpreted as the dimensionless speed of short waves and quantity $\sqrt{1-\gamma_A^2}$ as the dimensionless speed of long waves according to the linear dispersion analysis (see [2, 5]). For the FSE the linear dispersion analysis results in dispersion curves that have two branches – the acoustic and the optical branches. For the HE there exist only one branch (which corresponds to the acoustic branch of the FSE). The character of the dispersion for the HE is defined by quantity $\Gamma = 1 - \gamma_A^2 - \gamma_1^2$. If $\Gamma > 0$ we have the normal dispersion, if $\Gamma < 0$ the anomalous dispersion and for $\Gamma = 0$ the dispersionless case (see [5] for details).

We have solved the HE (3) and FSE (2) for $0.02 \leq \gamma_A^2 \leq 0.98$ and $0.02 \leq \gamma_1^2 \leq 0.98$ under localized initial conditions and periodic boundary conditions $U(X,0) = U_o \mathrm{sech}^2 B_o\left(X - \kappa\pi\right),\ U(X,T) = U(X + 2\kappa m\pi, T),\ m = 1, 2, \ldots$. For the amplitude and the width of the initial pulse we use the values $U_o = 1$ and $B_o = 0.5$. Initial phase speed is taken to be zero and $2\kappa\pi$ is the length of the spatial period. For the FSE (2) we take $\varphi(X,0) = 0$ and $\varphi_T(X,0) = 0$, i.e., we assume that at $T = 0$ the microdeformation and the corresponding velocity are zero. For numerical integration the discrete Fourier transform based pseudospectral method is applied [6].

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RESULTS AND DISCUSSION

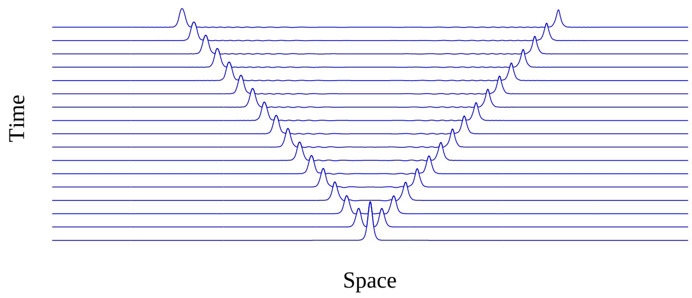

Figure 1. Solutions of the FSE, time-slice plot for $0 \leq T \leq 180$; $\gamma_A^2 = 0.54$, $\gamma_1^2 = 0.46$.

Under considered initial and boundary conditions and values of parameters two solitary waves (that propagate in opposite directions) emerge from the initial bell-shaped pulse (Fig. 1). Besides the main solitary pulses (dispersive) oscillating structures can be detected. In Fig. 2 we present two waveprofiles for $\gamma_A^2 = 0.54$ and $\gamma_1^2 = 0.46$ at the end of the integration interval $T_f = 180$ (parameters: $A = 1.170 \cdot 10^{11}$, $B = 3.816 \cdot 10^8$, $C = 2.574 \cdot 10^{10}$, $D = 5.0 \cdot 10^9$, $M = 5.0 \cdot 10^{10}$, $N = 1.0 \cdot 10^{11}$, $\rho = 8950$, $I = 4475$, $\delta = 0.0016$). The profile in the left panel corresponds to the HE and that in the right panel to the FSE. The values $\gamma_A^2 = 0.54$ and $\gamma_1^2 = 0.46$ result in the dispersionless case, i.e. $\Gamma = 1 - \gamma_A^2 - \gamma_1^2 = 0$. Therefore no oscillations can be detected for the solution of the HE. However, the situation is more complicated in case on the FSE, because now the dispersion curve has two branches. According to the acoustic branch the case is dispersionless, but according to the optical branch it is not.

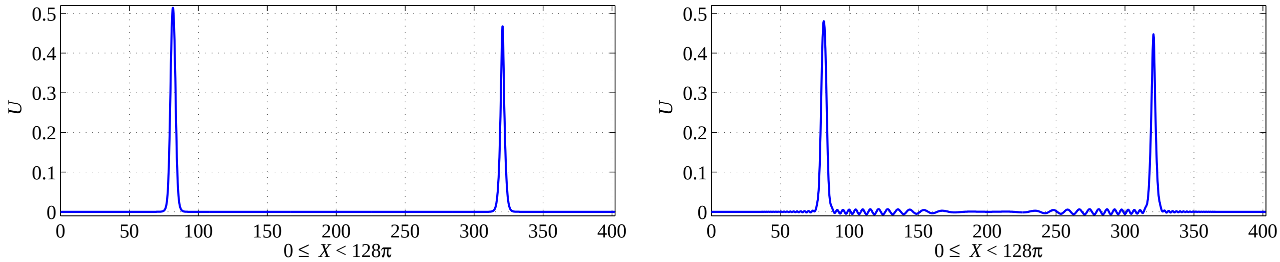

Figure 2. Solutions of the HE (the left panel) and the FSE (the right panel), $\gamma_A^2 = 0.54$, $\gamma_1^2 = 0.46$ and $T_f = 180$.

This fact is related to the existence of the oscillating structure between the pulses. Due to the latter formation of oscillations the FSE pulses are lower than the HE pulses. In the linear case (nonlinear parameters $M = N = 0$) the left and the right pulses are identical and the solution is symmetric about $X = \kappa\pi$. In the nonlinear case emerged pulses become asymmetric and the pulse that propagates to the right is lower than that of propagating to the left. The direction of the asymmetry of waveprofiles is dependent on the dispersion type and nonlinearity accelerates the emergence of asymmetry.

According to the linear dispersion analysis one can predict that solutions of the FSE and HE (i) have good concurrency in the domain where acoustic branches of dispersion curves of the FSE and HE differ less than 5 %, and (ii) behave differently in the normal and anomalous dispersion cases). These predictions are hold also for the nonlinear cases. However, nonlinearity introduces additional effects (asymmetry, sharpening of pulses etc.) not taken into account by (linear) dispersion analysis.

Analysis of interactions of emerged solitary waves demonstrates that the smaller the quantity $\gamma_A^2 \cdot |\Gamma|$, the closer the behaviour of the solutions of the HE and FSE to the solitonic behaviour.

More examples and information will be given during the presentation (see also [7]).

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