THREE DIMENSIONAL ANALYTICAL SOLUTION FOR BRAZILIAN TEST

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Summary This paper summarizes our recently obtained 3-D elasticity solution for Brazilian test or indirect tensile test. Typical testing setup recommended by ISRM and ASTM were used in our calculations. This new solution improves upon the widely adopted classical (2-D) solutions by Hertz (1883) and by Hondros (1959). For the recommended diameter to length ratio of 2, the 2-D solution can have an error of up to 15%. For diameter to length ratio of 2, the error in tensile strength may be up to 60%.

INTRODUCTION

Tensile strength is a very important index for describing brittle materials because it is more relevant to the failure of brittle solids than compressive strength. However, direct tensile test is very difficult to conduct without inducing any eccentric moments. The most popular indirect tensile strength test for rocks and concretes is the so-called Brazilian test (although this test was actually proposed by the Japanese first). It has been standardized by both International Society for Rock Mechanics (ISRM) and American Society for Testing and Materials (ASTM). The recommended height-to-diameter ratios (*H/D*) for both ISRM and ASTM standards are 0.5. Failure always occurs in brittle splitting along the planes between the applied line loads.

The analytical solution for a solid circular cylinder subjected to two diametral line loads was derived by Hertz in 1883 (p. 124, Timoshenko and Goodier, 1982). The main feature of this solution is that a uniform tensile stress is predicted on the vertical plane formed by joining the two line loads. Indeed, circular cylinders did fail in tension between these two line loads in all brittle materials. Not surprisingly, this simple and elegant solution has been adopted in the standard testing procedures of both ISRM and ASTM. Hondros (1959) extended the solution to the case of applied load being modeled as uniformly distributed strip loads. Both of these 2-D solutions are valid for either very long cylinders (plane strain condition) or very short cylinders (plane stress condition). However, the suggested *H/D* value in both ISRM and ASTM standards is 0.5 which is neither long nor short. The use of 2-D approximation is not fully justified. Indeed, it is more often found that the experimental results cannot be well described by the 2-D analytical solution (Yu et al., 2006).

Therefore, Yu et al. (2006) used the finite element method (FEM) to study the shape effect in the Brazilian test with 3D FEM. Numerical results show that for a fixed Poisson's ratio of 0.22 the tensile stress distribution along the compressed diameter and thickness is not uniform, and the tensile stress near the end surface of the specimen is higher than that of the inner part. It was also found that the 2-D solution by Hertz and Hondros (1959) is not accurate enough to calculate the tensile strength of rocks, especially for relatively thick cylinders. The main objective of this study is to obtain a three-dimensional (3-D) analytical solution for the indirect tensile test, and through this new solution to investigate the validity of 2-D solution in applying to indirect tensile tests.

METHOD OF SOLUTION

The method of solutions follows the displacement function approach (Chau and Wei, 2000). In particular, the two uncoupled differential equations for displacement functions are biharmonic equation and Laplace equation. They are obtained in cylindrical coordinate, and the general solutions of these two displacement functions are expressed in series solution consisting of Bessel functions, hyperbolic functions and trigonometric functions:

$$u_r = \frac{\partial^2 \Phi}{\partial r \partial z} + \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \qquad u_\theta = \frac{1}{r}\frac{\partial^2 \Phi}{\partial \theta \partial z} - \frac{\partial \Psi}{\partial r}, \qquad u_z = -[2(1-\nu)\nabla_1 \Phi + (1-2\nu)\frac{\partial^2 \Phi}{\partial z^2}], \qquad \nabla^4 \Phi = 0, \qquad \nabla^2 \Psi = 0 \tag{1}$$

$$\begin{aligned}\Phi = -A_0 \frac{z^3}{12G} - C_0 \frac{z}{4G} r^2 - \frac{1}{2G}\sum_{n=0}^{\infty}\Bigg\{ & H_{0n} r^{2n+2} z + \sum_{m=1}^{\infty}\frac{1}{\eta_m^3}[A_{mn} r \frac{\partial I_{2n}(\eta_m r)}{\partial r} + B_{mn} I_{2n}(\eta_m r)]\sin(\eta_m z) \\ & + \sum_{s=1}^{\infty}\frac{1}{\gamma_s^3}[C_{sn}\sinh(\gamma_s z) + D_{sn}\gamma_s z\cosh(\gamma_s z)] J_{2n}(\gamma_s r)\Bigg\}\cos(2n\theta)\end{aligned} \tag{2}$$

$$\Psi = -\frac{1}{2G}\sum_{n=0}^{\infty}\left\{ E_{0n} r^{2n} + \sum_{m=1}^{\infty}\frac{E_{mn}}{\eta_m^2} I_{2n}(\eta_m r)\cos(\eta_m z) + \sum_{s=1}^{\infty}\frac{F_{sn}}{\gamma_s^2}\cosh(\gamma_s z) J_{2n}(\gamma_s r)\right\}\sin(2n\theta) \tag{3}$$

where $\eta_m = m\pi / h$, $\gamma_s = \lambda_s / R$, and λ_s is the *s*-th root of $J'_{2n}(\lambda_s) = 0$ (i.e. derivative of the Bessel function of the first kind); $J_{2n}(x)$ and $I_{2n}(x)$ are the Bessel and modified Bessel functions of the first kind with order $2n$, respectively (Abramowitz and Stegun, 1965). The unknown constants have to be determined by the boundary conditions. In particular, the boundary conditions are traction free everywhere except for the radial stress within two narrow loading strips on both top and bottom of the curved surface. This loading boundary condition can be expanded in Fourier-Bessel expansion:

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$$\sigma_{rr} = -\frac{F}{\pi R} - \sum_{n=1}^{\infty} \frac{2F}{n\pi b} J_1\left(\frac{2nb}{R}\right) \cos 2n\theta \tag{4}$$

In our actual calculations, the infinite series are truncated to a finite number. If all m, n and s equal a finite integer q, the number of unknowns become $2(q+1)+6q(q+1)$. There would be exactly $2(q+1)+6q(q+1)$ algebraic equations for this system of equations from the boundary conditions. In our numerical calculations, 50 terms have been used.

MAXIMUN HOOP STRESS WITHIN THE CYLINDER

Figure 1 demonstrates that the maximum hoop stress roughly appears in the range: $0.7 < z/h < 0.8$ along the axis of the cylinder. The plots of hoop stress given in Fig. 1 are only for the geometric ratio $H/D = 0.5$ recommended by the ISRM and ASTM standards. The dotted line is the 2-D Hondros (1959) solution whereas the left diagram is for ISRM setup on rocks and the right diagram is for ASTM setup on asphalts. The maximum tensile near the end surface increases with ν while the tensile stress at center decreases with ν. The error of the 2-D solution is obvious.

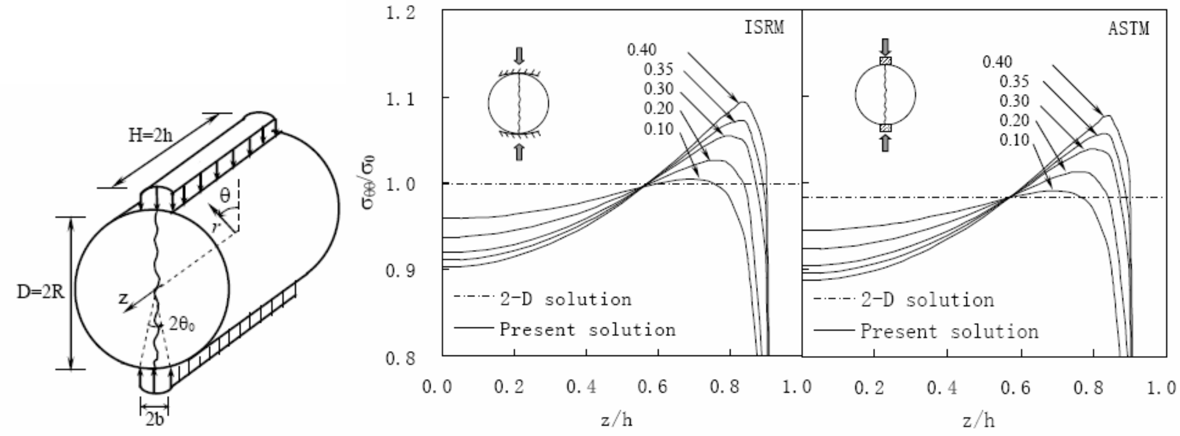

Figure 1. The normalized hoop stress versus the normalized distance z/h along the line $r/R = 0$ and $\theta = 0$ for various Poisson's ratio ν. The results for ISRM and ASTM contacts are shown on the left and right hand side respectively. The corresponding 2-D solution is also included as a center line, where $\sigma_0 = 2F/(\pi D)$ is the Hertz 2-D solution for the hoop stress

CONCLUSION

A 3-D analytical solution for solid cylinders under the indirect tensile test is obtained. The present solution can be considered as a 3-D counterpart of the classical 2-D solution obtained by Hertz in 1883 (see Timoshenko and Goodier, 1982) and Hondros (1959). The traction boundary conditions induced by the applied load from the compression machine through platens of concave shape for ISRM (1978) standard and through platens of flat surface for ATSM (2004) standard were considered by using Hertz contact. Once the traction boundary condition was determined from Hertz contact, the elastic stress in the finite solid cylinder is solved by using two displacement functions. The general solution forms of these displacement functions are expressed in series of Bessel functions, hyperbolic functions and trigonometric functions. By applying Fourier-Bessel series expansion technique, all the boundary conditions are satisfied exactly.

The current practice of using specimens of shape of $H/D = 0.5$, as suggested by ASTM (2004) and ISRM (1978), does not warrant the use of 2-D solution either the Hertz formula of 1883 (Timoshenko and Goodier, 1982) or the Hondros (1959) solutions. It seems that major revision is needed in the current code of practice to reflect the inaccurate prediction of indirect tensile strength using the 2-D solution. One simple way to remedy the problem is to introduce a correction factor (based on the present 3-D solution) to the classical 2-D solution as a function of shape of the cylinder and Poisson's ratio of the material.

In summary, the present study should provide a useful solution and theoretical basis in allowing us to better interpret the tensile strength of solids under indirect tensile test.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (Grant No. 11032003 and 10872033) and by a grant from The Hong Kong Polytechnic University (Project No. 1-BBZF). This work was also supported by the opening project of the State Key Laboratory of Explosion Science and Technology (KFJJ09-11).

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