STATICALLY ADMISSIBLE STRESS FIELDS IN LOOSE SAND HEAPS UNDER THE CLOSURE OF POLARIZED PRINCIPAL AXES

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Summary This study presents theoretical solutions of stress distributions in semi-infinite sand heap sloping at an angle of repose with loose condition. These solutions are categorized to statically admissible stress fields because static equilibriums and boundary conditions are satisfied and the limiting stress condition is not violated. Formulation under the stress closure of polarized principal axes with the self-similarity was achieved with its immediate consequence that the stress solutions can be expressed in the separated-variable form between its depth and gradient. The exact stress solutions agreed well with the past measurements for both prismatic and conical shapes.

BASIC ASSUMPTIONS AND EXACT STRESS SOLUTIONS

The stress distribution under self-weight loading in sand heaps is a basic problem in granular physics and geo-mechanics. Gravitating sand heaps are formed on the rigid base at their angle of repose ϕ. Two common geometries of the heap are prismatic and conical shapes. The stable shape of sand heaps on a horizontal plane without lateral supports is attributed by friction angle ϕ which is taken equivalent to angle of repose for loose deposition. This fundamental problem is seemingly simple but not easy to tackle; thus, it is proven to be a significant challenge in physics and mechanics [1].
Both prismatic and conical sand heap have a symmetry about the central axis as shown in Fig.1 and Fig.2, respectively; therefore, a rectangular (x,z) and cylindrical coordinate system (x,φ,z) with z-axis passing the symmetrical line is convenient to employ, where x is the distance measured from the center line, φ is the angular position measured on the horizontal plane around the central axis and z is the axial position measured from its apex. The configuration of stresses acting on the positive plane of each sectional element is also shown in Fig.1 and Fig.2. Stress components which are horizontal stress σ_x, vertical stress σ_z, shear stress τ_{xz} and hoop stress σ_φ (additional component for a conical geometry) must satisfy the following equilibrium conditions under self-weight loading with constant unit weight γ, i.e. referring to Eq.(1) and Eq.(2) for a prismatic shape, referring to Eq.(3) and Eq.(4) for a conical shape:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xz}}{\partial z}=0\,,\quad \frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \sigma_z}{\partial z}=\gamma \qquad (1),\ (2)$$

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xz}}{\partial z}+\frac{\sigma_x-\sigma_\varphi}{x}=0\,,\quad \frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{xz}}{x}=\gamma \qquad (3),\ (4)$$

Apart from equilibrium conditions, unknown stress requires additional equations to close the system of equation. Herein, Nadai's assumption (1963) [2] is adopted for characterizing angle of the major principal stress axis ω as shown in Figs.3 and 4 with relations to referenced coordinate system. The orientation of the major principal stress along any circle of constant radius traced around the apex simply converges toward the same point situated at the top of the circle; thus any straight line drawn from the pole to the bulk will intersect the circle at an angle equal to half of its central angle θ to the vertical plane.

$$\omega=\theta/2\,,\quad \tan\theta = x/z \qquad (5),\ (6)$$

As a result, the stress closure of polarized principal axes can be formulated in Eq.(7), using Eq.(5) and Eq.(6).

$$\sigma_x=\sigma_z-2\tau_{xz}/\tan 2\omega=\sigma_z-2\tau_{xz}\,z/x \qquad (7)$$

For conical shape, a supplementary equation is fulfilled by the hypothesis of hoop stress based on Levy which assumes $\sigma_\varphi=\sigma_x$. By following the assumptions used in the past studies [4], the characteristic of the problem is self-similar and have no inherent length scale. Thus, stress distributions appear in the similar geometric shape at all distances from the apex of sand heap.

$$s=(x/z)\tan\phi\,,\quad \sigma_x(x,z)=\gamma z\chi_x(s)\,,\quad \sigma_z(x,z)=\gamma z\chi_z(s)\,,\quad \tau_{xz}(x,z)=\gamma z\chi_{xz}(s) \qquad (8),\ (9),\ (10),\ (11)$$

Statically admissible stress solutions satisfying stress-free traction along the sliding plane are summarized below.
Prismatic loose sand heap [3]:

$$\frac{\sigma_x}{\gamma z}=\chi_x=\frac{1}{2}\frac{\tan\phi}{\cos^2\phi}\left(\left(3\phi+\left(2+\frac{1}{1+\cot^2\phi}\right)\cot\phi\right)-\left(3\cot^{-1}(s\cot\phi)+\left(2+\frac{1}{1+(s\cot\phi)^2}\right)s\cot\phi\right)\right) \qquad (12)$$

$$\frac{\sigma_z}{\gamma z}=\chi_z=\frac{1}{2}\frac{\tan\phi}{\cos^2\phi}\left(\left(\phi+\left(2-\frac{1}{1+\cot^2\phi}\right)\cot\phi\right)-\left(\cot^{-1}(s\cot\phi)+\left(2-\frac{1}{1+(s\cot\phi)^2}\right)s\cot\phi\right)\right) \qquad (13)$$

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$$\frac{\tau_{xz}}{\gamma z} = \chi_{xz} = \frac{1}{2}\frac{s}{\cos^2\phi}\left(\left(\cot^{-1}\left(s\cot\phi\right)+\frac{s\cot\phi}{1+\left(s\cot\phi\right)^2}\right)-\left(\phi+\frac{\cot\phi}{1+\cot^2\phi}\right)\right) \tag{14}$$

Conical loose sand heap [4]:

$$\frac{\sigma_x}{\gamma z} = \chi_x = \tan^4\phi\left(\left(\frac{8}{3}+4\cot^2\phi+\cot^4\phi\right)-\left(\frac{8}{3}+4\left(s\cot\phi\right)^2+\left(s\cot\phi\right)^4\right)\left(\left(1+\cot^2\phi\right)/\left(1+\left(s\cot\phi\right)^2\right)\right)^{3/2}\right) \tag{15}$$

$$\frac{\sigma_z}{\gamma z} = \chi_z = \tan^4\phi\left(\left(\frac{4}{3}+2\cot^2\phi+\cot^4\phi\right)-\left(\frac{4}{3}+2\left(s\cot\phi\right)^2+\left(s\cot\phi\right)^4\right)\left(\left(1+\cot^2\phi\right)/\left(1+\left(s\cot\phi\right)^2\right)\right)^{3/2}\right) \tag{16}$$

$$\frac{\tau_{xz}}{\gamma z} = \chi_{xz} = s\tan^3\phi\left(\left(\frac{2}{3}+\left(s\cot\phi\right)^2\right)\left(\left(1+\cot^2\phi\right)/\left(1+\left(s\cot\phi\right)^2\right)\right)-\left(\frac{2}{3}+\cot^2\phi\right)\right) \tag{17}$$

Experimental data [5] for vertical pressure underneath sand heaps with ϕ=33° filled by the spread source is considered due to the desired condition of homogeneous deposition. The theoretical solutions agreed well with the stress measurements normalized by γz, where z is the height of sand heaps, for both geometries as shown in Fig.5.

CONCLUSIONS

This study presents self-similar solutions of stress distribution for semi-infinite sand heap sloping at an angle of repose in loose condition for both prismatic and conical shapes based on the author's recent achievement on deriving exact stress solutions [3,4] using a set of assumptions used by Nadai (1963) [4].

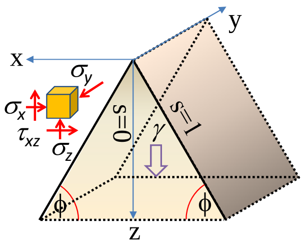

Fig.1 Geometry and configuration of a prismatic sand heap

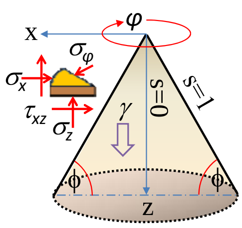

Fig.2 Geometry and configuration of a conical sand heap

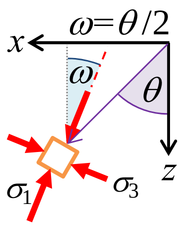

Fig.3 Description of Nadai's stress closure of polarized principal axes in sand heaps

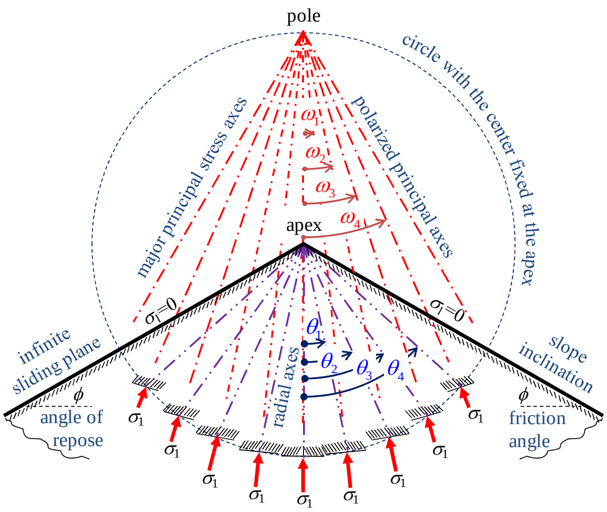

Fig.4 Diagram of the direction of major compression in semi-infinite symmetrical sand heap lying at angle of repose

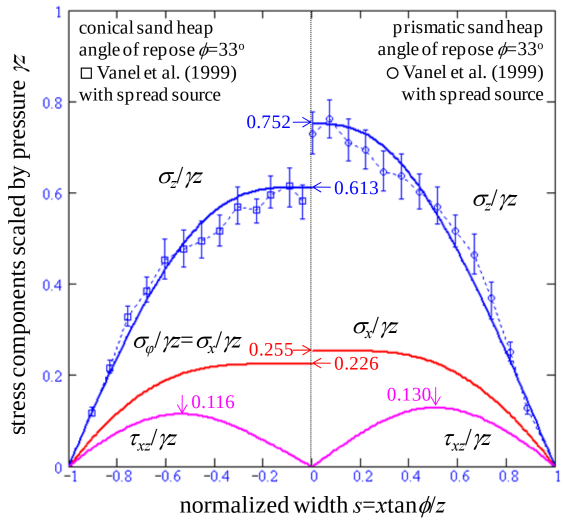

Fig.5 Comparisons with experimental data of sand heaps reported in Vanel et al. (1999)

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