SECULAR MOTIONS OF SPIN-DOWN TOPS AND PLANETS

Yutaka Shimomura
Department of Physics, Hiyoshi Campus, Keio University,
Yokohama, 223-8521, Japan

Summary Analytical solution is derived for the secular motions of spin-down tops, which keep contact with a fixed point. It predicts the change of the spin-axis orientation and precession rate when the spin slows down due to frictions, which agree well with the numerical simulation of the full Euler's angular-momentum equations. Especially, it finds their asymptotic limits when the spin rate goes to zero. Similar analytical solutions are also obtained for the secular motions of planets, whose oblatenesses are assumed to be the only factors to cause the moments by the gravity of Sun. According to them, their obliquities (the spin-axis orientations) are very stable for long time, as is true for the spinning tops.

SPINNIG TOPS

The motion of a spinning top is so attractive that many studies have been done so far, but some of its behaviours are yet to be well understood even now, as was true in the spinning-egg problem [1]. In this short paper, we think about one of such behaviours to find an analytical solution that describes the secular motions of spinning tops from the initial to the final state. Especially, we focus on the change of the spin-axis orientation and its asymptotic value in the limit of zero spin.

A spinning top considered here is an axisymmetric rigid body, whose principal momenta of inertia are $\mathrm{A, A, C}$ at O, the bottom end of the axis. The top spins under the gravity, with O always keeping contact with a fixed point, as shown in Fig.1 for example. Initially it spins so fast, then the precession rate Ω gradually increases as the spin rate n decreases due to the resisting moment of a couple of friction force, and as a result, the inclination angle θ (the angle between the vertical line and the spin axis of symmetry) increases very slowly.

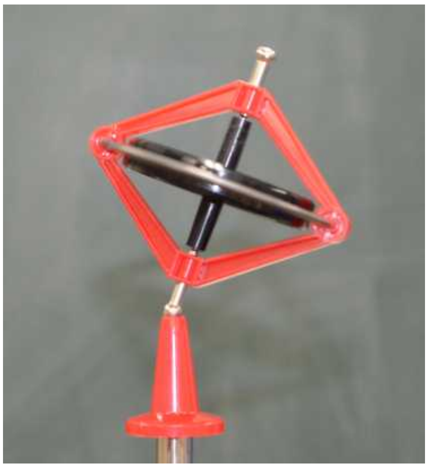

Figure 1
An example of a spinning top
with a fixed point of contact

ANALYTICAL SOLUTION

Now, let us assume that the inclination angle θ, the spin rate n, and the precession rate Ω change in time t slowly enough, for $d^2\theta/dt^2$ to be neglected in one of the Euler's angular-momentum equations. This assumption means that we observe secular motions filtering out such faster motions as the nutation, and leads to a balance

$$\mathrm{Cn} = \mathrm{A}\Omega\cos\theta + \frac{mgl}{\Omega}, \qquad (1)$$

where $m, g,$ and l are the mass of a top, the acceleration of gravity, and the distance between the point of contact and the centre of mass, respectively. The formula (1) may be called an 'extended gyroscopic balance' (EGB) by the following reason.

When the first term on the right-hand side of (1) is much smaller than the other two terms in (1), the balance (1) is reduced to a well-known formula ([2], for example)

$$\Omega = \frac{mgl}{\mathrm{Cn}}. \qquad (2)$$

And, when the second term on the right-hand side of (1) is much smaller than the other two terms in (1), we obtain

$$\mathrm{Cn} = \mathrm{A}\Omega\cos\theta, \qquad (3)$$

which is called a 'gyroscopic balance' and approximately holds in the spinning-egg problem [1]. Lastly, when the term on the left-hand side of (1) is much smaller than the two terms on the right-hand side, we get

$$\mathrm{A}\Omega^2\cos\theta = -mgl, \qquad (4)$$

which corresponds to the relation found in the dynamics of Euler's disk [3].

This EGB (1) fortunately decouples n from Ω and θ in the Euler's angular-momentum equations, and we find from one of them for Ω,

$$\Omega^2\sin^2\theta = \frac{2mgl}{\mathrm{A}}(\cos\theta_0 - \cos\theta) + \Omega_0^2\sin^2\theta_0\,, \qquad (5)$$

where θ_0 and Ω_0 are the initial values of θ and Ω, respectively. If θ is given, the formula (5) determines Ω, and the right-hand side of (1). So, the value of n, which is gradually decreases by the friction moment, determines θ

from the balance (1). Therefore the expressions (1) and (5) constitute the analytical solution for the secular motion of a spinning top.

RESULTS

The analytical solution (1) and (5) derives θ and Ω as functions of n for an example case, which is shown in Fig. 2. It is checked that these profiles agree precisely with those obtained by the numerical simulation of the full Euler's angular-momentum equations. From Fig.2, we find the asymptotic dimensionless values of θ and Ω in the zero limit of n are about 1.9 rad (> 90 degree) and 1.3, respectively. These results should be valid as long as the top keeps contact with a fixed point even when θ gets larger than 90 degree.

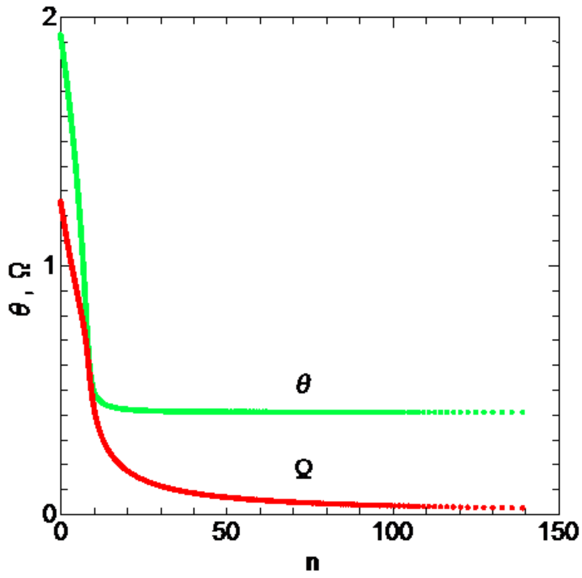

Figure 2
Analytical predictions for a spinning top:
θ & Ω versus n

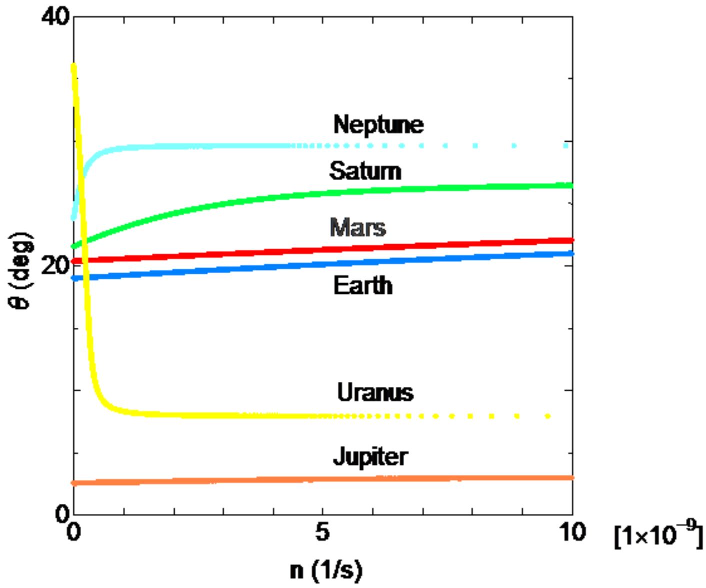

Figure 3
Analytical predictions for planets:
θ versus n

APPLICATION TO PLANETS

Here, let us think about large-scale tops, or, planets. We apply the analytical solution, which is derived by the same way as for the formula (1) and (5), to a rigid and oblate spheroid as a model of a planet. Here, we simply assume that only the oblateness of a planet produces the moments acting on its secular motions by the gravitation of Sun. Fig. 3 shows very small θ as a function of n for 6 planets, namely, Earth, Mars, Jupiter, Saturn, Uranus (90 degrees should be add to θ for this case), and Neptune. We notice that θ generally does not depend on n so much and that θ for all the planets except Uranus decreases as n goes to zero. Reminding the spins of planets are decelerated bit by bit by such effects as the tidal dissipation [4] or the core-mantle friction [5], this means that their obliquities are rather stable for long time.

CONCLUSIONS

Analytical solution for the secular motions of spinning tops that keep contact with a fixed point is obtained. It clarifies the secular change of their motions as the spin decreases due to frictions and determines the asymptotic limits of inclination angle and precession rate when the spin finally vanishes. The same study is applied to the secular motions of planets, whose oblatenesses are assumed to be the only factors to produce moments. As a result, their analytical solutions predict that the obliquities do not change significantly in the long time of spin-down processes, which is also the case for the spinning tops.

References

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