GRANULAR RHEOLOGY: FINE TUNED FOR OPTIMAL EFFICIENCY?

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Summary We solve a combination of two classical optimisation problems known as the Maximum Flow Problem and Minimum Cost Flow Problem to understand the relation between the contact network and the transmission of force through this network. We explore the connection between force transmission and flows which maximise the total amount of flow through the network, from source node to sink node, such that the total cost of the flow is minimised. Our ultimate goal is to quantify force transmission through various pathways in the contact network at a cost associated with the energy dissipated at the contacts. Here we take the first steps toward this goal and present the basic concepts and preliminary results to illustrate some aspects of the method.

INTRODUCTION

Think of the contact network in a quasistatically deforming granular material as a flow network. Flow networks are ubiquitous in everyday life and are characterised by some commodity being passed from one node to another in the network: product components through assembly lines, fluid or gas through pipelines, current through electrical networks, information through communication networks, vehicles through roadways etc. [1]. A recurring question of great practical interest is: how can we ensure optimal efficiency of flow in these networks subject to certain constraints? Here we explore the extent to which various aspects of granular rheology (e.g. force transmission) might be optimised through a given contact network, in a sequence of consecutive complex contact networks representing the deforming material throughout loading history. The aim of this paper is to use network optimisation techniques to understand how a granular material reacts optimally to externally applied loads. The analysis proceeds in two steps. First, we construct a flow network at each strain state, which includes source, sink and intermediate nodes based on the contact network. Second, we determine the maximum flow that can be transmitted through the network from source to sink under minimal cost.

METHODOLOGY

Consider a dense sample of particles under plane strain compression subject to constant confining pressure from the lateral boundaries. At each strain state, we construct a *directed* complex network from the contact network, wherein the nodes represent the particles and two oppositely directed arcs the interparticle contact. The sense of direction from a node i to a node j represents the direction of a flow that can be transmitted through the link or arc (i, j). As forces in granular materials are transmitted through interparticle contacts, we are interested in paths through the contact network that optimises force transmission. As an example, we consider the discrete element simulation in [2]. At each strain state, we find the maximum value of flow that can be transmitted through the contact network from the top wall to the bottom wall where the sample is compressed. We solve a maximum flow problem, where the problem is defined as follows.

Let $\mathcal{N} = \{1, 2, \dots, N, N+1, N+2\}$ be the set of nodes in the directed complex network, where N denotes the number of particles in the sample. Suppose each node in the directed network *supplies* a certain amount of flow, i.e. $s_i =$ supply of node $i \in \mathcal{N}$. There are three types of nodes: *source* ($s_i > 0$), *sink* ($s_i < 0$) and *transshipment node* or *intermediate node* ($s_i = 0$). We assume that node $N+1$ is source and node $N+2$ is sink while all other nodes are transshipment nodes. Thus, $s_i = 0$, $i = 1, \dots, N$, $s_{N+1} \in (0, \infty)$, and $s_{N+2} \in (-\infty, 0)$.

Let $\mathcal{A}$ be the set of directed links (or arcs) of the network, where (i, j), $(j, i) \in \mathcal{A}$ if there is a contact between particle i and particle j, $(N+1, i) \in \mathcal{A}$ if particle i is in contact with the source, and $(i, N+2) \in \mathcal{A}$ if particle i is in contact with the sink. Here, arc (i, j) is a directed edge from node i to node j.

For each $(i, j) \in \mathcal{A}$, let f_{ij} be the flow along arc (i, j). An ordered set of values $f = \{f_{ij}, (i, j) \in \mathcal{A}\}$ is a *flow* if it satisfies the following constraints:

$$0 \le f_{ij} \le \alpha_{ij}, \qquad (i, j) \in \mathcal{A}, \tag{1}$$

and

$$\sum_{j:\,(i,j)\in\mathcal{A}} f_{ij} - \sum_{j:\,(j,i)\in\mathcal{A}} f_{ji} = s_i, \qquad i \in \mathcal{N}, \tag{2}$$

where α_{ij} denotes the capacity along arc (i, j). Note that the contact network is an undirected network. Thus, the flow through the interparticle contact between particles i and j can be measured by $|f_{ij} - f_{ji}|$.

Maximum flow problem and minimum cost flow problem

Find an ordered set of values $f = \{f_{ij}, (i, j) \in \mathcal{A}\}$ to maximise s_{N+1} subject to constraints (1) and (2). Here, s_{N+1} is the total amount of flow f that is transmitted through the network from source to sink. This maximum flow problem can be solved by Ford-Fulkerson method [3]. Its solution comprises the flow that transmits the maximum amount of flow, $f_{\max}$,

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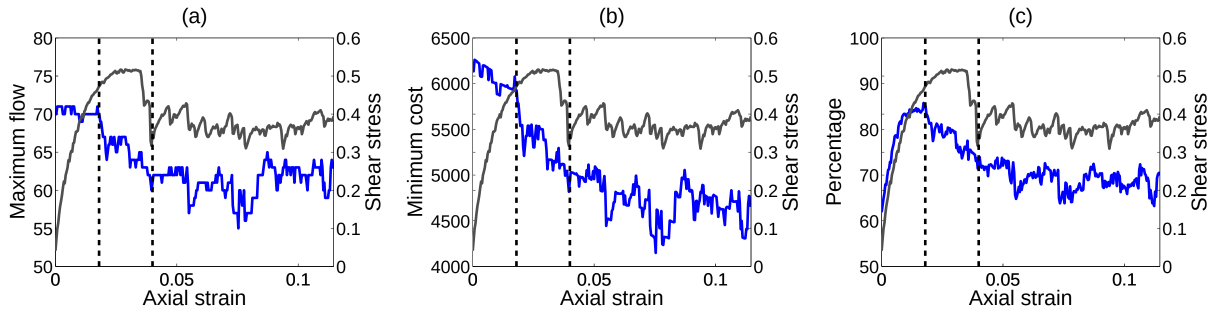

Figure 1. (a) The evolution with axial strain of the maximum flow $f_{\max}$ transmitted through the network from source to sink. (b) The evolution with axial strain of the minimum cost that transmits flow $f_{\max}$ through the network from source to sink. (c) The evolution with axial strain of the percentage of contacts in the force chain network transmitting nonzero flow in the minimum cost flows.

from source to sink through the directed network. In a granular material, energy is dissipated at the contacts (e.g. through friction). This dissipation can be modelled as a cost to the flow f_{ij} through arc (i, j). Let $E(f) = \sum_{(i,j)\in\mathcal{A}} c_{ij} f_{ij}$ be the cost of flow f, where c_{ij} denotes the associated cost through arc (i, j). To find the minimum cost when the total amount of flow transmitted from source to sink is maximum $f_{\max}$, we solve the minimum cost flow problem which involves finding an ordered set of values $f = \{f_{ij}, (i, j) \in \mathcal{A}\}$ that minimises the cost $E(f)$ subject to $s_{N+1} = f_{\max}$ and constraints (1) and (2). This minimum cost flow problem can be solved by the network simplex method [3].

RESULTS

For each strain state, we solve the combination of the maximum flow problem and the minimum cost flow problem outlined above to explore a possible interplay between structural development (here contact topology) and system dynamics in the DEM simulation of [2]. Of interest is how the solution to this network optimisation problem relates to known pathways in the contact network, e.g. force chain network. We consider one example where the top wall acts as the source and the bottom wall acts as the sink. Furthermore, for each $(i, j) \in \mathcal{A}$, we choose $c_{ij} = 1$ and

$$\alpha_{ij} = \begin{cases} \infty, & \text{if } i = N+1 \text{ or } j = N+2 \\ 1, & \text{otherwise.} \end{cases}$$

Here, the contacts are uniformly weighted, hence the maximum flow and minimum cost flow problems use only information on the contact topology. As shown in Figure 1(a), the maximum flow problem captures the three distinct regimes (indicated by black dashed lines) of flow initially observed in [2]. We emphasise here the three distinct regimes of flow in [2] were established not just from information on contacts but also particle kinematics. We observe the maximum flow $f_{\max}$ remains essentially constant during the initial stages of the strain-hardening regime during which deformation is found to be globally affine in [2]. Then $f_{\max}$ suddenly decreases rapidly as nonaffine deformation and dissipation develop and intensify, a trend which continues (modulo fluctuations) to the end of the strain-softening regime when the shear band becomes fully developed. $f_{\max}$ fluctuates about a near constant value in the large strain or 'critical state' regime. The minimum cost to transmit the maximum flow from source to sink, shown in Figure 1(b), is decreasing during the strain-hardening and shear band development regimes. Of interest is the relationship between the contacts in the minimum cost flow network and those between particles in the force chain network. Note that the minimum cost flow is not unique: there are multiple paths through the contact network that can transmit the maximum flow from source to sink with the same minimum cost. What we are interested in are those paths that are always used to transmit nonzero flow in the minimum cost flows. Figure 1(c) shows the axial strain evolution of the percentage of contacts in the force chain that are always used to transmit nonzero flow in the minimum cost flows. During the strain-softening regime, the percentage of contacts in the force chain transmitting nonzero flow in the minimum cost flows increases dramatically from 61% to 86%, drops to 72% when the shear band is fully developed, and then fluctuates at about 70% in the critical state regime. On average, throughout the loading history, around 72% of such contacts are also in the force chain network. This suggests that force chains play a significant – albeit not exclusive – role when transmitting maximum flow from source to sink with the minimum cost.

References

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