WEIGHT FUNCTION FOR AN ELLIPTICAL CRACK IN PIEZOELECTRIC MEDIA

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Summary In the present study an integral equation method has been used for the first time to obtain the weight functions for an elliptical crack embedded in an infinite transversely isotropic piezoelectric medium, the plane of the crack being parallel to the plane of isotropy of the medium. Some results of approximate weight functions together with both type of intensity factors are given.

INTRODUCTION

It is observed that almost nothing has been done as far as evaluation of weight functions for three-dimensional cracks in piezoelectric media are concerned. In the present study the integral equation method developed by Roy and his coworkers (see Roy and Chatterjee [1] and Roy and Saha [2]) has been adopted to solve the problem under consideration. Finally, expressions for the mechanical and electric weight functions, for an elliptical crack embedded parallel to the plane of isotropy of a transversely isotropic piezoelectric media, are given in a form suitable for computational purpose. Some results of approximate weight functions and intensity factors are also given as example.

FORMULATION OF THE PROBLEM

Equations governing the three-dimensional piezoelectric media in absence of body forces and free charges, the piezoelectric stress constitutive relations and the mechanical stress-displacement and electric field-potential relations are given as follows:

$$\begin{aligned} \sigma_{ij,j} &= 0 \qquad \text{and,} \qquad D_{i,i} = 0 \\ \sigma_{ij} &= c_{ijkl}\epsilon_{kl} - e_{kij}E_k \qquad \text{and,} \qquad D_i = e_{ikl}\epsilon_{kl} + \zeta_{ik}E_k \\ \epsilon_{ij} &= \tfrac{1}{2}(u_{i,j} + u_{j,i}) \qquad \text{and,} \qquad E_i = -\phi_{,i} \end{aligned} \tag{1}$$

where σ_{ij}, D_i are mechanical stress and electric displacement components; ϵ_{ij}, E_i are mechanical strain and electric field components; c_{ijkl}, e_{ijk}, ζ_{ij} are elastic stiffness, piezoelectric and dielectric constants; u_i are displacement components and ϕ is the electric potential. In equations (1) $i, j, k, l = 1, 2, 3$ each.

Boundary conditions

Let the elliptic crack, embedded parallel to the plane of isotropy of the medium, occupy the region

$$S : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1, \;\; z = 0. \tag{2}$$

For a prescribed pair of applied mechanical and electrical point-loads of identical magnitude $\delta(x - x_0)\delta(y - y_0)$ acting normally in opposite directions to the crack faces, the boundary conditions on the crack plane are as follows:

$$\sigma_{zx}(x,y,0) = \sigma_{zy}(x,y,0) = 0; \;\; \left.\begin{array}{c} \sigma_{zz}(x,y,0) = -\delta(x-x_0)\delta(y-y_0) \\ D_z(x,y,0) = \delta(x-x_0)\delta(y-y_0) \end{array}\right\} \forall (x,y) \in S; \;\; \left.\begin{array}{c} u_z(x,y,0) = 0 \\ \phi(x,y,0) = 0 \end{array}\right\} \forall (x,y) \notin S. \tag{3}$$

The systems of algebraic equations

Following Wang and Huang [3] and Saha and Saha [4]one obtains a set of coupled integral equations which are further reduced following Roy and Saha [2] to obtain the following system of algebraic equations:

$$\begin{aligned} &\forall s = 0, 1, 2, \ldots, \infty, \; (n+s) \text{ even,} \\ &\sum_{n=0}^{s-2} I^C_{n,s} \begin{pmatrix} \mathsf{K}^n_{m+\frac{s-n}{2}} \\ \mathsf{T}^n_{m+\frac{s-n}{2}} \end{pmatrix} + I^C_{s,s} \begin{pmatrix} \mathsf{K}^s_m \\ \mathsf{T}^s_m \end{pmatrix} + \sum_{n=s+2}^{2m+s} I^C_{n,s} \begin{pmatrix} \mathsf{K}^n_{m-\frac{n-s}{2}} \\ \mathsf{T}^n_{m-\frac{n-s}{2}} \end{pmatrix} = \\ &\frac{2^m (m)! \left(2m+s+\frac{3}{2}\right) H}{a(2m+1)!!} r_0^s \left(1 - r_0^2\right)^{1/2} \cos s\theta_0 P_m^{\left(s,\frac{1}{2}\right)} \left(1 - 2r_0^2\right) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{aligned} \tag{4}$$

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and, another similar system with K,T replaced by $\overline{\mathsf{K}},\overline{\mathsf{T}}$ and $\cos(.)$ by $\sin(.)$. Here,

$$\left(\begin{array}{cc} \mathsf{K}_r^n & \overline{\mathsf{K}}_r^n \\ \mathsf{T}_r^n & \overline{\mathsf{T}}_r^n \end{array}\right) = \left(\begin{array}{cc} M_1 & M_2 \\ N_1 & N_2 \end{array}\right)\left(\begin{array}{cc} -E_r^n & -\overline{E}_r^n \\ V_r^n & \overline{V}_r^n \end{array}\right);\ \left[\psi_n,\overline{\psi}_n;\theta_n,\overline{\theta}_n\right](t) = \sum_{j=0}^{\infty}\left[V_j^n,\overline{V}_j^n;E_j^n,\overline{E}_j^n\right]P_j^{\left(n+\frac{1}{2},0\right)}\left(1-2t^2\right). \tag{5}$$

All other symbols are explained in Roy and Saha [2] and Saha and Saha [4].

WEIGHT FUNCTIONS

For a piezoelectric medium there will be a mechanical weight function $W_I^M(\phi;r_0,\theta_0)$ and an electric weight function $W_I^E(\phi;r_0,\theta_0)$. For any fixed N, the N th approximation of the weight functions are obtained in a suitable form as follows:

$$\left(\begin{array}{c} W_I^{M(N)} \\ W_I^{E(N)} \end{array}\right)(\phi;r_0,\theta_0) = \frac{2\sqrt{1-r_0^2}}{\pi abHB}\left(\frac{b}{a}\right)^{1/2}\left(a^2\sin^2\phi+b^2\cos^2\phi\right)^{1/4}\left(\begin{array}{cc} M_1 & M_2 \\ N_1 & N_2 \end{array}\right)\left(\begin{array}{c} -M_2-N_2 \\ M_1+N_1 \end{array}\right)\times$$
$$\sum_{m=0}^{N}\left[\sum_{k=0}^{m}\sum_{j=0}^{m}P_{m-j}^{2j}\left\{\frac{C_{2k,2j}^{c(2m)}}{\triangle_{2m}}\cos 2j\theta_0\cos 2k\phi+\frac{C_{2k,2j}^{s(2m)}}{\overline{\triangle}_{2m}}\sin 2j\theta_0\sin 2k\phi\right\}+\right.$$
$$\left.\sum_{k=0}^{m}\sum_{j=0}^{m}P_{m-j}^{2j+1}\left\{\frac{C_{2k+1,2j+1}^{c(2m+1)}}{\triangle_{2m+1}}\cos(2j+1)\theta_0\cos(2k+1)\phi+\frac{C_{2k+1,2j+1}^{s(2m+1)}}{\overline{\triangle}_{2m+1}}\sin(2j+1)\theta_0\sin(2k+1)\phi\right\}\right]. \tag{6}$$

where $C_{2k,2j}^{c(2m)}$ is the cofactor of $I_{2k,2j}^c$ in the $(m+1)\times(m+1)$ matrix $M_{2m} = \left(\begin{array}{cccc} I_{0,0}^c & I_{2,0}^c & \cdots & I_{2m,0}^c \\ I_{0,2}^c & I_{2,2}^c & \cdots & I_{2m,2}^c \\ \vdots & \vdots & \vdots & \vdots \\ I_{0,2m}^c & I_{2,2m}^c & \cdots & I_{2m,2m}^c \end{array}\right)$ and $\triangle_{2m} = \det(M_{2m})$. Similarly $C_{2k+1,2j+1}^{c(2m+1)}$ is the cofactor of $I_{2k+1,2j+1}^c$ in the $(m+1)\times(m+1)$ matrix M_{2m+1} whose elements are now $I_{2k+1,2j+1}^c$, $(k=0,1,\ldots,m$ and $j=0,1,\ldots,m)$ and $\triangle_{2m+1}=\det(M_{2m+1})$. On the other hand, $C_{2k,2j}^{s(2m)}$ is the cofactor of $I_{2k,2j}^s$ in the $m\times m$ matrix $\overline{M}_{2m}$ whose elements are $I_{2k,2j}^s$, $(k=1,2,\ldots,m$ and $j=1,2,\ldots,m)$ with $\overline{\triangle}_{2m}=\det(\overline{M}_{2m})$ and $C_{2k+1,2j+1}^{s(2m+1)}$ is the cofactor of $I_{2k+1,2j+1}^s$ in the $(m+1)\times(m+1)$ matrix $\overline{M}_{2m+1}$ whose elements are now $I_{2k+1,2j+1}^s$, $(k=0,1,\ldots,m$ and $j=0,1,\ldots,m)$ and $\overline{\triangle}_{2m+1}=\det(\overline{M}_{2m+1})$. Other symbols may be found in Saha and Saha [4].

Once the weight functions are determined the mechanical stress and electric displacement intensity factors, $K_I^M(\phi)$ and $K_I^E(\phi)$ respectively, for prescribed arbitrary mechanical load $L^M(r_0,\theta_0)$ and electric load $L^M(r_0,\theta_0)$ are determined as

$$\left(\begin{array}{c} K_I^M(\phi) \\ K_I^E(\phi) \end{array}\right) = ab\int_0^{2\pi}\int_0^1\left(\begin{array}{c} W_I^M(\phi;r_0,\theta_0)L^M(r_0,\theta_0) \\ W_I^E(\phi;r_0,\theta_0)L^E(r_0,\theta_0) \end{array}\right)dr_0d\theta_0. \tag{7}$$

EXAMPLES

• 0th Approximation of the Weight Function at (r_0,θ_0) :

(a) penny-shaped crack: $W_I^{M(0)} = W_I^{E(0)} = \frac{\sqrt{1-r_0^2}}{\pi^2 a^{3/2}}\left[3+10r_0\cos(\phi-\theta_0)\right]$

(b) elliptic crack: $W_I^{M(0)} = W_I^{E(0)} = \frac{\sqrt{1-r_0^2}\left(\sin^2\phi+\frac{b^2}{a^2}\cos^2\phi\right)^{1/4}}{\pi a\sqrt{b}}\left[\frac{3}{I_{0,0}^c}+\frac{5r_0}{I_{1,1}^c}\cos\theta_0\cos\phi+\frac{5r_0}{I_{1,1}^s}\sin\theta_0\sin\phi\right]$

• Stress Intensity Factor for constant loading 'p' using equation (7):

(a) penny-shaped crack: $K_I^M = K_I^E = \frac{2\sqrt{b}p}{\pi}$; (b) elliptic crack: $K_I^M = K_I^E = \frac{2p\sqrt{b}\left(\sin^2\phi+\frac{b^2}{a^2}\cos^2\phi\right)^{1/4}}{I_{0,0}^c}$

CONCLUSION

The mechanical and electric weight functions for an elliptical crack embedded in a transversely isotropic piezoelectric medium parallel to the plane of isotropy has been evaluated for the first time and given in a form that is suitable for computation. Some results of weight functions and intensity factors are also given using the expression. The expression may be used to give the intensity factors for non-polynomial loadings also.

References

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