THE SCALED BOUNDARY FINITE ELEMENT METHOD FOR 2D FRACTURE ANALYSIS OF PIEZOELECTRIC MATERIALS

Chao Li [a)], Hou Man, Chongmin Song & Wei Gao
School of Civil and Environmental Engineering, University of New South Wales, Sydney, 2052, Australia

Summary In this paper, the scaled boundary finite element method (SBFEM) is extended to 2D fracture analysis of piezoelectric materials. The stress and electric displacement (extended stress) intensity factors are directly determined from the scaled boundary finite element solution for the singular extended stress field without requiring any asymptotic solution. A numerical example with impermeable electric crack face boundary conditions is presented to show the accuracy and efficiency of this method.

INTRODUCTION

Piezoelectric materials have interrelated mechanical and electrical properties, which leads to the widespread applications of these materials. However, due to the brittleness and complex electromechanical loads, cracks are easily developed. Therefore, it is necessary to understand and investigate the fracture behaviour of piezoelectric materials. In order to assess the reliability of piezoelectric structures, the determination of extended stress intensity factors plays a key role to evaluate the severity of the singular stress and electric displacement at the crack tips.
In this paper, SBFEM is extended to evaluate the extended stress intensity factors of piezoelectric materials in 2D situation. This method is a semi-analytical method developed by Song and Wolf [1], which is more applicable to solve fracture problems of anisotropic materials [2]. Numerical result shows this method is an efficient tool to solve the fracture problem of piezoelectric materials accurately.

FUNDAMENTAL EQUATIONS OF PIEZOELECTRIC MATERIALS

The standard linear piezoelectric constitutive equations in stress-charge form (plain strain condition) are given as

$$\boldsymbol{\sigma} = \boldsymbol{c\varepsilon} - \boldsymbol{e}^{\mathrm{T}}\boldsymbol{E}, \qquad \boldsymbol{D} = \boldsymbol{e\varepsilon} + \boldsymbol{\epsilon E} \tag{1}$$

where $\boldsymbol{\sigma}$ denotes the stress tensor, $\boldsymbol{D}$ the electric displacement vector, $\boldsymbol{\varepsilon}$ the strain tensor, $\boldsymbol{E}$ the electric field vector, $\boldsymbol{c}$ the elastic constants, $\boldsymbol{e}$ the piezoelectric constants, $\boldsymbol{\epsilon}$ the permittivity. The mechanical strains and electric fields are

$$\varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}), \qquad E_i = -\phi_{,i} \tag{2}$$

where u_i is the displacement vector, ϕ the electric potential scalar.
The elastostatic and electrostatic equilibrium equations in the domain for the forces and the electric charges are

$$\sigma_{ij,j} + f_{bi} = 0, \qquad D_{i,i} - q_b = 0 \tag{3}$$

with f_{bi} the mechanical body forces and q_b electric body charges.

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The extended displacement (displacement and electric potential) at a point in the domain are denoted as $\{\bar{u}\} = \left[u_x,\ u_y,\ \phi\right]^T$. The extended strain (strain and electric field) are denoted as $\{\bar{\varepsilon}\} = \left[\varepsilon_{xx},\ \varepsilon_{yy},\ \varepsilon_{xy},\ -E_x,\ -E_y\right]^T$. The extended stress are given as $\{\bar{\sigma}\} = \left[\sigma_{xx},\ \sigma_{yy},\ \tau_{xy},\ D_x,\ D_y\right]^T$. The scaled boundary finite element equation in the scaled boundary coordinate $(\xi,\ \eta)$ is derived as

$$[E^0]\xi^2\{\bar{u}(\xi)\}_{,\xi\xi} + ([E^0] - [E^1] + [E^1]^T)\xi\{\bar{u}(\xi)\}_{,\xi} - [E^2]\{\bar{u}(\xi)\} = 0 \tag{4}$$

where $[E^0]$, $[E^1]$ and $[E^2]$ are the element coefficient matrices in SBFEM [1]. The internal nodal force and electric displacement along radial lines are

$$\left\{\bar{q}\{\xi\}\right\} = [E^0]\xi\{\bar{u}(\xi)\}_{,\xi} + [E^1]^T\{\bar{u}(\xi)\} \tag{5}$$

The extended displacement solution of SBFEM is obtained analytically as a matrix function [3]

$$\{\bar{u}(\xi)\} = \sum_{i=1}^{N-1}[\Psi_i^u]\,\xi^{-[S_i]}\{c_i\} + [\Psi_N^u]\{c_N\} \tag{6}$$

where $\{S\}$ is a quasi-upper triangular matrix, $\{c\}$ is integration constant. The real parts of the eigenvalues of a diagonal block [S] are in [-1, 0], which leads to a stress and electric displacement singularity [4]. When the polar coordinates (r, θ) are introduced, the singular extended stress is expressed as

$$\left\{\bar{\sigma}^{(s)}(r,\theta)\right\} = \left[\Psi_{\sigma L}^{(s)}(\theta)\right](\tfrac{r}{L})^{-[S^{(s)}]-[I]}\left\{c^{(s)}\right\} \tag{7}$$

with $\left[\Psi_{\sigma L}^{(s)}(\theta)\right]$ the singular extended stress modes at the characteristic length L (superscript (s) for singular) and $[I]$ identity matrix. The extended stress intensity factors $\{K(\theta)\} = [K_I(\theta),\ K_{II}(\theta),\ K_{IV}(\theta)]$ at angle θ are defined by the singular extended stress $\left[\sigma_{\theta\theta}^{(s)},\ \tau_{r\theta}^{(s)},\ D_\theta^{(s)}\right]$

[a)] Corresponding author. Email: chao.li3@student.unsw.edu.au

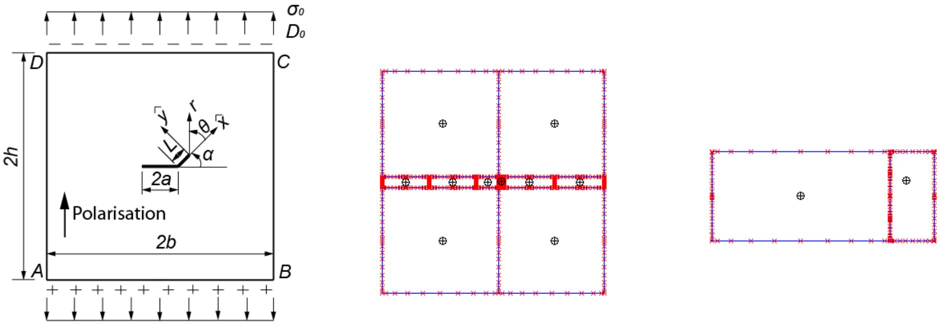
$$\{\bar{\sigma}^{(s)}(r,\theta)\} = \frac{1}{\sqrt{2\pi L}}\left[\Psi_{\sigma L}^{(s)}(\theta)\right]\left(\frac{r}{L}\right)^{-[S^{(s)}]-[I]}\left[\Psi_{\sigma L}^{(s)}(\theta)\right]^{-1}\{K(\theta)\} \tag{8}$$

From Eqs. (7) and (8), the extended stress intensity factors are given

$$\{K(\theta)\} = \sqrt{2\pi L}\left[\Psi_{\sigma L}^{(s)}(\theta)\right]\{c^{(s)}\} \tag{9}$$

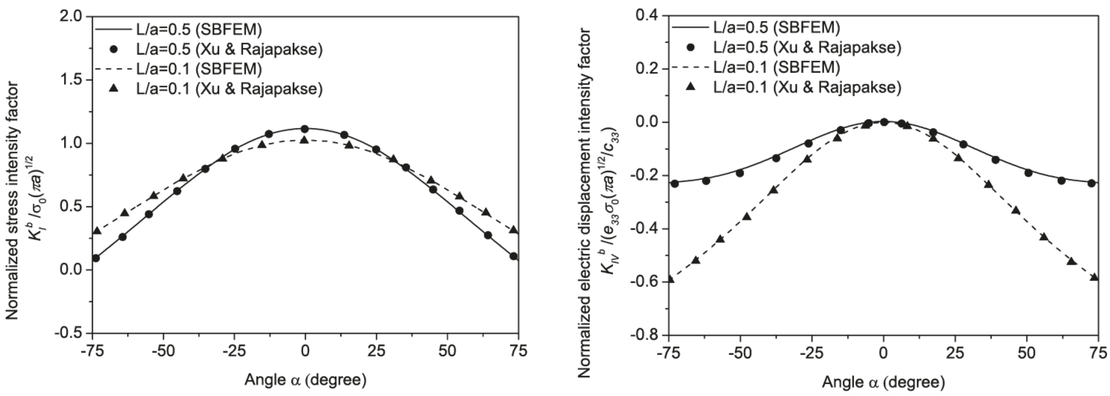
NUMERICAL EXAMPLE

A PZT4 infinite plate with a non-straight crack is subjected to a tension load σ_0 shown is Figure 1(a). The main crack of the length $2a$ with a branched crack of the length L is in the plate. The branch length $L = 0.5a$ or $L = 0.1a$ are modelled. This infinite plate is truncated into a square with the side length $h = 20a$ and $b = h$. The poling direction is perpendicular to the main crack. The branched crack is denoted by the angle α, with respect to the main crack. Figure 1(b) shows the scaled boundary finite element mesh for the cracked plate. The scaling centers are placed at the crack tips or the centers of the subdomains, which are indicated by the markers ⊕. Impermeable electric crack face boundary conditions are used in this example. Figure 1(c) shows elements in the vicinity of crack. Figure 2(a) and 2(b) plot the stress intensity factor K_I normalized with $\sigma_0\sqrt{\pi a}$ and electric displacement intensity factor K_{IV} normalized with $(e_{33}\sigma_0\sqrt{\pi a}/c_{33})$ respectively. Present SBFEM results agree with the exact solutions in [5] very well.



(a) (b) (c)

Figure 1 Infinite plate with a branched crack: (a) geometry; (b) SBFEM mesh; (c) elements in the vicinity of crack



(a) (b)

Figure 2 (a) Normalized stress intensity factors K_I; (b) Normalized electric displacement intensity factors K_{IV}

CONCLUSIONS

SBFEM is extended to calculate the extended stress intensity factors of piezoelectric materials. The extended stress intensity factors have also been determined from the singular extended stress analytically obtained in SBFEM without requiring any asymptotic solution. Numerical example shows that the scaled boundary finite element method can solve the fracture problems of piezoelectric materials accurately and simply.

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