WEIGHT FUNCTIONS OF A CRACK IN A TWO-DIMENSIONAL MICROPOLAR SOLID

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Summary
The plane-strain and antiplane-strain model problems on a crack in an unbounded micropolar solid are analyzed. In both cases one out of three modes is uncoupled, and the other two are coupled. For a semi-infinite crack, the models reduce to a scalar and an order-2 vector Wiener-Hopf problems. In the plane-strain case, the weight functions are derived by quadratures. For a finite antiplane-strain crack, the weight functions are found in a series form through the solution of an infinite system of linear algebraic equations.

INTRODUCTION AND THE MAIN RESULTS

The effect of couple-stresses on the stress concentration at the tip of a finite crack under conditions of plane strain subject to uniform uni-axial tension was studied by Sternberg and Muki [1]. By using the method of Fredholm integral equations they found that when the couple-stress parameter $l \to 0$, the stress intensity factor does not approach that obtained for the elastic case. Both cases, the constrained couple-stress theory and micropolar elasticity, were analyzed by Atkinson and Leppington [2] for a semi-infinite and a finite crack under conditions of plane strain. In the case of a semi-infinite crack, the problem was reduced to a system of two Wiener-Hopf equations. The system was decoupled, and the solution was found by quadratures. The problem of a finite crack was solved by the method of matched asymptotic expansions. For both theories, they showed that when a certain parameter, s_1, tends to zero, then the energy release rate from a crack tip tends to the classical elastic value, while the stress intensity factor does not. That parameter was taken as the couple-stress parameter l (also used in [1]) for the constrained couple-stress theory, and the parameter $s_1 = \gamma(\kappa+\mu)[\kappa(\kappa+2\mu)]^{-1}$, for the micropolar case. Here, γ, κ and μ are material constants. Another interesting limiting case $s \to \infty$ ($s = l$ in the the constrained couple-stress theory, and $s = s_1$ in micropolar elasticity) is hard to implement if the solutions derived in [1] and [2] are used, and it was discussed neither in [1], nor in [2]. We analyze the problem for a semi-infinite crack subject to loading $\tau_{yx} = p_1(x)$, $\sigma_y = p_2(x)$, $m_{yz} = p_3(x)$. In contrast to [1], [2], in our derivation we do not employ the Mindlin formulation [3] in terms of the generalized Airy stress function. Instead, we apply the Fourier transform directly to the governing equilibrium equations of micropolar elasticity. This ultimately gives a representation of the solution that is easy to analyze when $s_1 \to \infty$. We show that if the elastic parameter $\kappa \to 0^+$, and γ remains finite and non-zero (in this case $s_1 \to +\infty$), then the equilibrium equations, the stress-strain relations and the solution reduce to the classical equations and formulas of elasticity. For a semi-infinite crack, we use the Wiener-Hopf method and derive exact formulas for the stress intensity factors and the four weight functions, the functions $W_{I,j}$ and $W_{VI,j}$ ($j = I, VI$) associated with the modes I (σ_y) and VI (m_{yz}), respectively (for the standard mode-I, II (τ_{yx}) and III (τ_{yz}) cracks see the diagram [4], p.59; for the other modes see Fig. 1). We show that if the elastic parameter $\kappa \to 0^+$, and γ remains finite and non-zero (in this case $s_1 \to +\infty$), then the equilibrium equations, the stress-strain relations and the solution of the Wiener-Hopf problem reduce to the classical equations and formulas of elasticity. As $\kappa \to 0^+$, the weight functions $W_{I,I}$, W_{II} and $W_{VI,VI}$ tend to $-\sqrt{2(\pi\xi)^{-1}}$ that is the mode-I and mode-II weight function of elasticity. The other two functions $W_{I,VI}$ and $W_{VI,I}$ vanish. If $\gamma \to 0^+$ and $\kappa/\gamma \to 0^+$, then $W_{VI,VI} \to 0$, and the only nonzero weight functions are $W_I = W_{I,I}$ and W_{II}. The asymptotic analysis is illustrated by numerical results.

Li and Lee [5] analyzed the antiplane-strain problem for a finite crack subject to the boundary conditions $\tau_{yz} = p(x)$, $m_{yx} = 0$, $m_{yy} = 0$, $|x| < a$, $y = 0^\pm$, and reduced the problem for the whole plane with a crack to the one for a half-pane subject to the boundary conditions $u_z = 0, \phi_x = 0$, $\phi_y = 0$ outside the crack. We show that these boundary conditions are not correct, and must be replaced by $u_z = 0, m_{yx} = 0$, $\phi_y = 0$. For a semi-infinite crack in the case of loading $\tau_{yz} = p_1(x)$, $m_{yx} = p_2(x)$, $m_{yy} = p_3(x)$, $0 < x < a$, $y = 0^\pm$, the antiplane-strain problem reduces to a scalar and an order-2 vector Wiener-Hopf problems. We solve the mode-V scalar Wiener-Hopf problem in closed form and find the mode-V weight function by two quadratures. For a finite mode-III,IV crack, the problem is equivalent to a system of two singular integral equations with the Cauchy and logarithmic kernels. The system is solved by the method of orthogonal polynomials. We derive the weight functions in a series form by using the generating function of the Chebyshev polynomials $U_n(x)$. The coefficients of the series solve a certain infinite system of linear algebraic equations. Again, as in the plane-strain case, if $\kappa \to 0^+$, then the mode-III and IV weight functions associated with the crack tips $x = 0$ and $x = a$ tend to the classical elastic weight functions.

THE METHOD OF SOLUTION

A plane-strain semi-infinite crack

The problem to be analyzed is that of a semi-infinite crack $\{0 < x < \infty\ y = 0^\pm\}$ in an otherwise unbounded micropolar solid. The crack faces are subjected to plane-strain loading $\tau_{yx} = p_1(x), \sigma_y = p_2(x), m_{yz} = p_3(x)$, $0 < x < \infty$, $y = 0^\pm$. If $-\infty < x \le 0$, the displacements u_x and u_y and the microrotation ϕ_z are continuous through the line $y = 0$,

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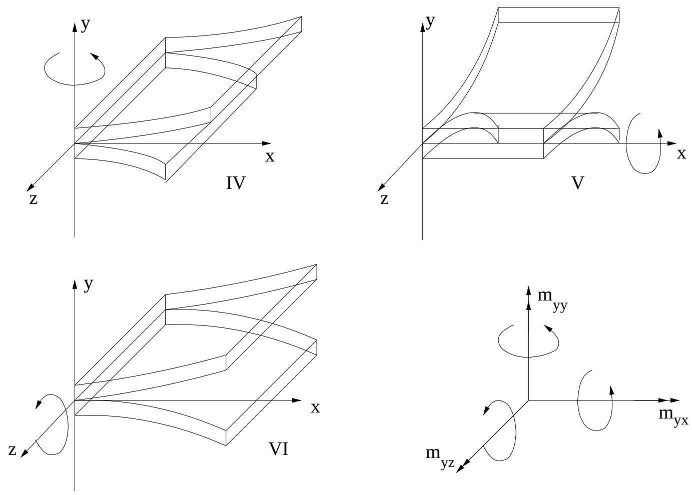

Figure 1. The mode-IV, V and VI cracks.

and they are discontinuous if $0 < x < \infty$. By applying the Fourier transform to the the equilibrium equations written in terms of the displacements u_x and u_y and the microrotation ϕ_z we ultimately arrive at the vector Wiener-Hopf equation $\mathbf{\Phi}^+(\zeta) = d(\zeta)G(\zeta)[\mathbf{\Phi}^-(\zeta) + \hat{\mathbf{p}}^+(\zeta)]$, $\zeta \in L$, where L is the real axis,

$$G(\zeta) = \begin{pmatrix} \frac{r}{|\zeta|} & 0 & 0 \\ 0 & \frac{r}{|\zeta|} & -i\zeta\left(1 - \frac{r}{|\zeta|}\right) \\ 0 & i\zeta\left(1 - \frac{r}{|\zeta|}\right) & \kappa_0^2 - \zeta^2\left(1 - \frac{r}{|\zeta|}\right) \end{pmatrix},$$

$r = \sqrt{\zeta^2 + \rho^2}$, ρ and κ_0 are certain parameters, $d(\zeta)$ is a certain function, and $\hat{\mathbf{p}}^+(\zeta)$ is the one-sided Fourier transform of the loading vector. The vectors $\mathbf{\Phi}^-(\zeta)$ and $\mathbf{\Phi}^+(\zeta)$ are given by $\mathbf{\Phi}^-(\zeta) = \int_{-\infty}^0 ||\tau_{yx}, \sigma_y, m_{yz}||(x, 0)e^{i\zeta x}dx$, $\mathbf{\Phi}^+(\zeta) = \int_0^\infty ||u_x(x, 0^+) - u_x(x, 0^-), u_y(x, 0^+) - u_y(x, 0^-), \phi_z(x, 0^+) - \phi_z(x, 0^-)||e^{i\zeta x}dx$. The mode-II Wiener-Hopf problem and the order-2 vector problem for the coupled modes I and VI are solved exactly by the factorization method. This solution is employed to find the weight functions associated with the stress intensity factors

$$K_{II} = \int_0^\infty W_{II}(\xi)p_1(\xi)d\xi, \quad \begin{pmatrix} K_I \\ K_{VI} \end{pmatrix} = \int_0^\infty \begin{pmatrix} W_{I,I}(\xi) & W_{I,VI}(\xi) \\ W_{VI,I}(\xi) & W_{VI,VI}(\xi) \end{pmatrix} \begin{pmatrix} p_2(\xi) \\ p_3(\xi) \end{pmatrix} d\xi.$$

An antiplane-strain finite crack

The problem to be solved consists in finding a solution $\{w, \phi_x, \phi_y\}$ of the system of governing equations of micropolar antiplane elasticity [5] subject to the boundary conditions $\tau_{yz} = p_1(x)$, $m_{yx} = p_2(x)$, $m_{yy} = p_3(x)$, $0 < x < a, y = 0^\pm$. The displacement u_z and the microrotations $\phi_x(x, y)$ and $\phi_y(x, y)$ are continuous through the line $y = 0$ as $x < 0$ or $x > a$. If $a = \infty$, then the problem reduces to a vector Wiener-Hopf equation with the matrix coefficient

$$G(\zeta) = \begin{pmatrix} G_{11}(\zeta) & 0 & iG_{13}(\zeta) \\ 0 & G_{22}(\zeta) & 0 \\ -iG_{13}(\zeta) & 0 & G_{33}(\zeta) \end{pmatrix}.$$

The structure of the matrix $G(\zeta)$ indicates that the mode V is decoupled, while the other two modes, III and IV, are coupled. The same is valid if the crack is finite. For a semi-infinite mode-V crack we find a closed-form solution and an exact formula for the associated weight function. The other two modes are coupled, and the matrix does not admit a closed-form factorization by the methods available in the literature. For a finite crack, we reduce the problem to a system of two singular integral equations and solve it approximately by the method of orthogonal polynomials. The weight functions are found in a series form in terms of the solution of an infinite system of linear algebraic equations.

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