INTERFACIAL CRACKS IN BI-MATERIAL SOLIDS: STROH FORMALISM AND SKEW-SYMMETRIC WEIGHT FUNCTIONS

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Summary The problem of a semi-infinite plane interfacial crack between two dissimilar anisotropic media is reduced to a matrix eigenvalue problem by means of Stroh formalism. The non-trivial singular solutions of the homogeneous problem for the crack are the weight functions. Using this Stroh formulation, both symmetric and skew-symmetric component of the weight functons are obtained by the analytical solution of the homogeneous equations, produced in closed form and accompanied by some examples of application.

GEOMETRY OF THE MODEL

A semi-infinite plane static crack propagating at the interface between two dissimilar anisotropic elastic materials with asymmetric self-balanced loading acting upon the crack faces is considered. The forces are applied outside a neighbourhood of the crack tip and are assumed to vanish at the infinity, the body forces are assumed to be zero. A Cartesian coordinates system with the origin at the crack tip is defined, the geometry of the model is shown in Fig.(1).

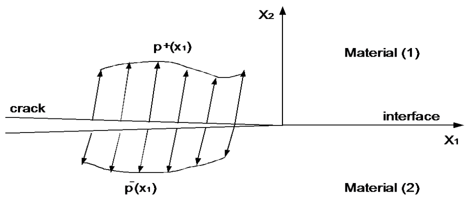

Figure 1. Geometry of the model.

STROH REPRESENTATION FOR DISPLACEMENTS AND STRESS FIELDS

Using the Stroh formulation of the balance equations [3], the linear elasticity problem for both the upper and lower half-plane can be reduced to a matrix eigenvalues problem. By means of this approach, the derivative of the displacement $\mathbf{u}_{,1}(x_1, x_2)$ and the traction $\boldsymbol{\mathcal{T}}(x_1, x_2) = (\sigma_{21}, \sigma_{22})^T$ for both half-planes can be written as follows:

$$\boldsymbol{\mathcal{T}}(x_1, x_2) \;=\; \mathbf{Bg}(\mathbf{z}) + \overline{\mathbf{Bg}(\mathbf{z})} \tag{1}$$

$$\mathbf{u}_{,1}(x_1, x_2) \;=\; \mathbf{Ag}(\mathbf{z}) + \overline{\mathbf{Ag}(\mathbf{z})} \tag{2}$$

where $\mathbf{A}$ and $\mathbf{B}$ are constant matrices depending by materials parameters for which general expressions are reported in references [2, 3], $\mathbf{g}(\mathbf{z})$ is a vector of analytic functions with components $g_j(x_1 + \mu_j x_2)$, and μ_j are complex eigenvalues of the Stroh eigensystem assumed with positive imaginary parts. According to the property enunciated by Suo [2], if $g_j(z_j)$ is an analytic function of $z_j = x_1 + \mu_j x_2$ in the upper half-plane (or in the lower half-plane) for one μ_j, where μ_j is a complex number with positive imaginary parts, it is analytic for any μ_j. This property makes possible reducing the analysis to the single complex variable. The traction (1) can be expressed in both upper and lower half-planes in terms non-homogeneous Riemann-Hilbert problem:

$$\mathbf{h}^{+}(z) - \mathbf{h}^{-}(z) = \boldsymbol{\mathcal{T}}(x_1, x_2), \tag{3}$$

If we assume the zero traction condition on the crack faces and the continuity of the tractions at the interface ahead the crack front ($\boldsymbol{\mathcal{T}}^{+} = \boldsymbol{\mathcal{T}}^{-} = \boldsymbol{\mathcal{T}}(x_1)$), and remembering that we are considering a crack running on the negative semi-axis, at the interface we have:

$$\mathbf{h}^{+}(x_1, 0^{+}) - \mathbf{h}^{-}(x_1, 0^{-}) \;=\; \boldsymbol{\mathcal{T}}(x_1) \quad \text{for} \quad x_1 > 0 \tag{4}$$

$$\mathbf{h}^{+}(x_1, 0^{+}) - \mathbf{h}^{-}(x_1, 0^{-}) \;=\; 0 \qquad \text{for} \quad x_1 < 0 \tag{5}$$

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Expriming the analytic function $\mathbf{h}(z)$ by means of the Plemelj formula, and using the integral definition of the Heaviside step function, we define the Fourier transform of $\mathbf{h}^{\pm}(z)$ respect to x_1. Substituing these expressions in the trasforms of (1) and (2), we obtain th Fourier transform respect to x_1 of the physical displacements:

$$\hat{\mathbf{u}}^{+}(\xi, x_2 = 0^{+}) = \Big\{ \frac{1}{2\xi}(\mathbf{Y}^{(1)} - \overline{\mathbf{Y}}^{(1)}) - \frac{1}{2|\xi|}(\mathbf{Y}^{(1)} + \overline{\mathbf{Y}}^{(1)}) \Big\} \hat{\boldsymbol{\mathcal{T}}} \tag{6}$$

$$\hat{\mathbf{u}}^{-}(\xi, x_2 = 0^{-}) = \Big\{ \frac{1}{2\xi}(\mathbf{Y}^{(2)} - \overline{\mathbf{Y}}^{(2)}) + \frac{1}{2|\xi|}(\mathbf{Y}^{(2)} + \overline{\mathbf{Y}}^{(2)}) \Big\} \hat{\boldsymbol{\mathcal{T}}} \tag{7}$$

Where $\mathbf{Y} = i\mathbf{A}\mathbf{B}^{-1}$ and the superscripts $^{(1)}$ and $^{(2)}$ denote the quantities related to the upper and lower half-planes, respectively.

SYMMETRIC AND SKEW-SYMMETRIC WEIGHT FUNCTIONS

The special singular solution $\mathbf{U}$ for the interfacial cracks are defined as solution of the elasticity problem where the crack is placed along the positive semi-axis, and the singulat traction $\boldsymbol{\Sigma}$ possesses non-zero values for $x_1 < 0$ (see [1, 4] for details). Consequently, we impose free traction contition for $x_1 > 0$ and traction continuity at the interface for $x_1 < 0$ and obtain a Riemann-Hilbert problem specular to the physical one:

$$\mathbf{h}^{+}(x_1, 0^{+}) - \mathbf{h}^{-}(x_1, 0^{-}) = 0 \qquad \text{for} \quad x_1 > 0 \tag{8}$$

$$\mathbf{h}^{+}(x_1, 0^{+}) - \mathbf{h}^{-}(x_1, 0^{-}) = \boldsymbol{\Sigma}(x_1) \quad \text{for} \quad x_1 < 0 \tag{9}$$

The Fourier transforms of singular displacements at the interface $\hat{\mathbf{U}}^{+}(\xi, x_2 = 0^{+})$ and $\hat{\mathbf{U}}^{-}(\xi, x_2 = 0^{-})$ are determined by means of the same approach used for the physical problem, based on Plemelj reprentation of $\mathbf{h}(z)$, and their expressions are identical to physical displacements (6) and (7), with $\hat{\boldsymbol{\Sigma}}$ in place of $\hat{\boldsymbol{\mathcal{T}}}$. Symmetric and skew-symmetric weight functions in Fourier space are derived respectively as the difference and the average of these singular fields [1]:

$$[\hat{\mathbf{U}}](\xi) = \hat{\mathbf{U}}^{+}(\xi) - \hat{\mathbf{U}}^{-}(\xi) = \frac{1}{|\xi|} \Big\{ i \, \mathrm{sign}(\xi) \, \mathrm{Im}(\mathbf{Y}^{(1)} - \mathbf{Y}^{(2)}) - \mathrm{Re}(\mathbf{Y}^{(1)} + \mathbf{Y}^{(2)}) \Big\} \hat{\boldsymbol{\Sigma}}(\xi) \tag{10}$$

$$\langle \hat{\mathbf{U}} \rangle(\xi) = \frac{\hat{\mathbf{U}}^{+}(\xi) + \hat{\mathbf{U}}^{-}(\xi)}{2} = \frac{1}{2|\xi|} \Big\{ i \, \mathrm{sign}(\xi) \, \mathrm{Im}(\mathbf{Y}^{(1)} + \mathbf{Y}^{(2)}) - \mathrm{Re}(\mathbf{Y}^{(1)} - \mathbf{Y}^{(2)}) \Big\} \hat{\boldsymbol{\Sigma}}(\xi) \tag{11}$$

Using the continuity of the traction across the interface the skew-symmetric weight function can be expressed in th form:

$$\langle \hat{\mathbf{U}} \rangle(\xi) = \boldsymbol{\mathcal{A}}[\hat{\mathbf{U}}](\xi) + \frac{i}{\xi} \boldsymbol{\mathcal{B}} \hat{\boldsymbol{\Sigma}}(\xi), \tag{12}$$

Where the diagonal matrix $\boldsymbol{\mathcal{A}}$ and the off-diagonal matrix $\boldsymbol{\mathcal{B}}$ are

$$\boldsymbol{\mathcal{A}} = \frac{1}{2} \, \mathrm{Re}(\mathbf{Y}^{(1)} - \mathbf{Y}^{(2)}) \Big(\mathrm{Re}(\mathbf{Y}^{(1)} + \mathbf{Y}^{(2)}) \Big)^{-1} \tag{13}$$

$$\boldsymbol{\mathcal{B}} = \frac{1}{2} \, \mathrm{Im}(\mathbf{Y}^{(1)} + \mathbf{Y}^{(2)}) - \boldsymbol{\mathcal{A}} \, \mathrm{Im}(\mathbf{Y}^{(1)} - \mathbf{Y}^{(2)}) \tag{14}$$

The (10) and (12) are general expressions for the Fourier transform of symmetric and skew-symmetric weight functions for a semi-infinite plane interfacial crack between two dissimilar anisotropic elastic materials.

CONCLUSIONS

General expressions for the Fourier transform of symmetric and skew-symmetric weight functions for a semi-infinite plane interfacial crack between two dissimilar anisotropic elastic materials have been derived, making the inversion of these trasforms the explicit representations for the weight functions are obtained [1, 4]. These expressions are valid for any kind of anisotropic media and can be used for studying several interfacial cracks problems evaluating the matrices $\mathbf{A}, \mathbf{B}, \mathbf{Y}$ for the considered materials. In particular, the derived weight functions are crucial for the calculation of stress intensity factor corresponding to an asymmetric loading applied on the crack faces, and of higher order terms of the asymptotic expressions of the stress near the crack tip, performed by means of Betti fundamental identity [1, 4].

References

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