A VARIATIONAL APPROACH TO THERMAL FRACTURE

Blaise Bourdin*[a)], Corrado Maurini**
*Department of Mathematics and Center for Computation & Technology, Louisiana State University, Baton Rouge, LA, USA.
**UPMC and CNRS Univ Paris 6, UMR 7190, Institut Jean Le Rond d'Alembert, 4 Place Jussieu, F-75005, Paris, France.

Summary We revisit a set of now classical experiments dealing with crack propagation in plates of brittle materials cooled along there edges. We propose a model based on Francfort and Marigo's variational approach to fracture which we implemented using a regularized model. We present quantitative comparison between numerical simulations and experiments.

VARIATIONAL MODEL FOR THERMAL FRACTURE

In the recent years, the variational approach to fracture has be the scope of intense investigations (see [4] and references within). In some situations, numerical simulations have been compared with closed form solutions, providing elements of *verification* of the model. A first attempt at a *validation* of the approach, in the context of directional drying of colloidal suspensions is presented in [9]. It relies on the indirect measurement of an inelastic strain and focusses on a situation where the cross-sectional crack geometry can be devised by a purely static model. Here, we apply the variational framework of Francfort and Marigo [6] to a now classical series of thermally driven fracture experiments and provide quantitative comparison between numerical simulations and experimental and analytical results from the literature [8, 7, 1, 2, 10, 11].

We consider an ideally brittle material with Young's modulus E, Poisson ratio ν linear thermal expansion coefficient α, and fracture toughness k occupying a thin rectangular slab of dimension $l_x \times l_y$, brought to a constant temperature θ_0 then cooled along one or several of its faces. As we are concerned in situations where the overall domain and the cracks length is large compared to the thickness of the plate, we represent the domain by a two-dimensional rectangle Ω and make the plain stresses hypothesis. We also assume that the cracks geometry has no impact on heat transfer. The temperature field is computed by solving a transient heat equation or given in closed form, depending on the geometry and boundary conditions. We account for the inelastic thermal strains in the variational framework by modifying the bulk term in the total energy. More precisely, at any given time t, the deformation field $u(t)$ and crack set $\Gamma(t)$ are given by the global minimizers under proper displacement boundary conditions and the crack growth hypothesis $\Gamma(t) \supset \Gamma(s)$ for all $s < t$ of the total energy

$$\mathcal{E}_{\theta(t)}(u, \Gamma) := \int_\Omega W(\mathrm{e}(u), \theta(t))\, dx + k\ell(\Gamma), \tag{1}$$

where $W(\epsilon, \theta) = \frac{1}{2}\mathbf{A}(\epsilon - \alpha\theta \mathrm{I}_2) : (\epsilon - \alpha\theta\mathrm{I}_2)$ is the bulk energy potential, $\mathrm{e}(u)$ denotes the linearized strain (the symmetrized gradient of u), $\mathbf{A}$ is the materials Hooke's law, I_2 is the two-dimensional identity matrix and ":" denotes the dot product of two matrices. No a priori hypothesis on the cracks geometry or topology are made. Instead, the crack geometry is derived from the unilateral minimization principle. Following the now classical approach originally presented in [3], we introduce a small regularization parameter ε with the dimension of a length, a secondary variable v representing the crack set, and the regularized functional

$$\mathcal{E}_{\varepsilon,\theta(t)}(u, v) := \int_\Omega (v^2 + o(\varepsilon)) W(\mathrm{e}(u), \theta(t))\, dx + \frac{3k}{8} \int_\Omega \frac{1-v}{\varepsilon} + \varepsilon |\nabla v|^2\, dx. \tag{2}$$

Following [5] for instance, it is possible to show that as $\varepsilon \to 0$, $\mathcal{E}_{\varepsilon,\theta(t)}$ converges to $\mathcal{E}_{\theta(t)}$ in the sense of Γ–convergence, which in turns implies that the minimizers of $\mathcal{E}_{\varepsilon,\theta(t)}$ converge to that of $\mathcal{E}_{\theta(t)}$. The regularized functional (2) is discretized using finite elements and an alternate minimzations algorithms is used.

COMPARISON BETWEEN NUMERICAL SIMULATIONS AND EXPERIMENTS

A basic dimensional analysis shows that this problem depends on a single parameter homogeneous to a length $l_0 = \frac{k}{E(\alpha\Delta\theta)^2}$, where $\Delta\theta$ represents the magnitude of the temperature shock applied at the edges. In particular when the domain is large enough and the temperature profile can be computed explicitly, the average crack spacing d as a function of the distance from the cooled edge a are linked in [2] by the scaling law $\frac{a l_0}{d^2} = C_1 \exp\left(-C_2 \frac{a}{d}\right)$,where C_1 and C_2 are two positive constants. Figure 1 shows the final cracks configuration and scaling law for several numerical simulations on domains with different sizes and aspect ratio. The domain geometry for experiments in [1, 7] is represented by red and green boxes, while that of the numerical simulations is shown in blue. Cracks distribution data from the literature are displayed with the scaling law (dark line) and compared to our simulations showing a very good agreement for many order of magnitude of scale parameter l_0. The same formalism can be used in three dimensions. In this case, cracks self-organize into polygonal cells reminiscent of cooling lava or drying starch. Cell merging is qualitatively similar to the two–dimensional parking phenomenon.

[a)] Corresponding author. E-mail: bourdin@lsu.edu

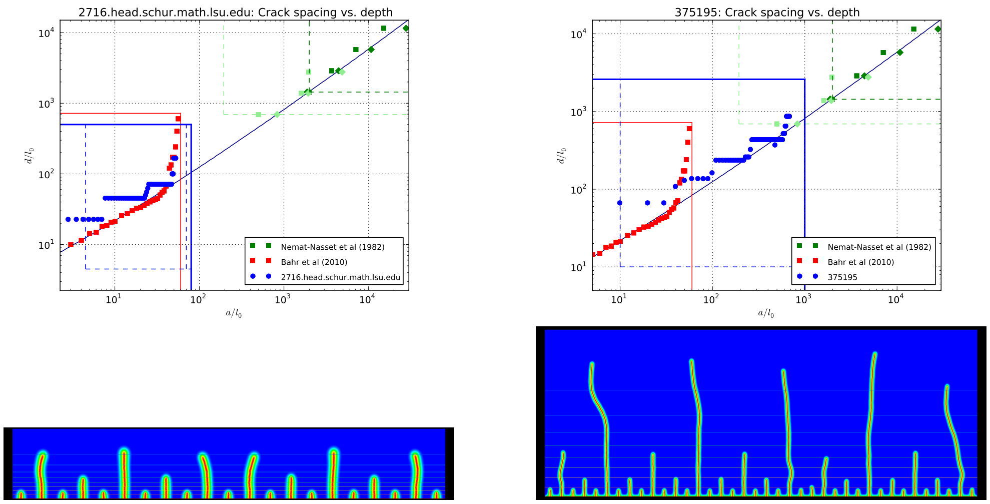

Figure 1. Crack geometry and scaling for experiments on a domain of size $500l_0 \times 80l_0$ (left) and $2,600l_0 \times 1,000l_0$ (right).

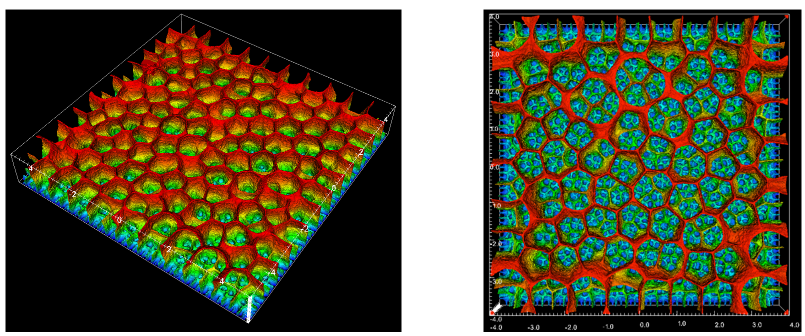

Figure 2. In the three-dimensional setting, our approach identifies complex crack geometries.

Acknowledgments.

B.Bourdin work was supported in parts by the National Science Foundation under Grant No. DMS-0909267. Some numerical experiments were performed using resources provided by TACC at the University of Texas under the NSF–Teragrid Resource Allocation TG-DMS060014N. The authors gratefully acknowledges the funding of the French National Research Council (CNRS) for a PICS bilateral exchange program. C.Maurini was partially supported by ANR program T-Shock OTP J11R087 and a grant of the University Pierre et Marie Curie EMERGENCES-UPMC.

References

[1] H. A. Bahr, G. Fischer, and H. J. Weiss. Thermal-shock crack patterns explained by single and multiple crack propagation. *Journal of Materials Science*, 21(8):2716–2720, 08 1986.

[2] H.-A. Bahr, H.-J. Weiss, U. Bahr, M. Hofmann, G. Fischer, S. Lampenscherf, and H. Balke. Scaling behavior of thermal shock crack patterns and tunneling cracks driven by cooling or drying. *J. Mech. Phys. Solids*, 58(9):1411–1421, 2010.

[3] B. Bourdin, G.A. Francfort, and J.-J. Marigo. Numerical experiments in revisited brittle fracture. *J. Mech. Phys. Solids*, 48(4):797–826, 2000.

[4] B. Bourdin, G.A. Francfort, and J.-J. Marigo. The variational approach to fracture. *J. Elasticity*, 91(1-3):1–148, 2008.

[5] A. Braides. *Approximation of Free-Discontinuity Problems*. Number 1694 in Lecture Notes in Mathematics. Springer, 1998.

[6] G.A. Francfort and J.-J. Marigo. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids.*, 46(8):1319–1342, 1998.

[7] J.F. Geyer and S.Nemat-Nasser. Experimental investigations of thermally induced interacting cracks in brittle solids. *Int. J. Solids Structures*, 18(4):137–356, 1982.

[8] L. M. Keer, S. Nemat-Nasser, and A. Oranratnachai. Unstable growth of thermally induced interacting cracks in brittle solids: Further results. *International Journal of Solids and Structures*, 15(2):111 – 126, 1979.

[9] C. Maurini, B. Bourdin, G. Gauthier, and V. Lazarus. Crack patterns in directional drying: numerical simulations using a variational approach and experiments. Submitted, 2011.

[10] Y. Shao, X. Xu, S. Meng, G. Bai, C. Jiang, and F. Song. Crack patterns in ceramic plates after quenching. *J. Am. Ceram. Soc.*, 93(10):3006–3008, 2010.

[11] Y. Shao, Y. Zhang, X. Xu, Z. Zhou, W. Li, and B. Liu. Effect of crack pattern on the residual strength of ceramics after quenching. *J. Am. Ceram. Soc.*, 94(9):2804–2807, 2011.