THE WEIGHT FUNCTION FOR CRACKS IN THREE-DIMENSIONAL QUASICRYSTALS

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Summary By using the generalized Stroh formulation, we analyse the generalized 2D problem of an infinite 3D QC medium with a crack. The general solutions are presented in explicit and closed form when the medium is subjected to the remote uniform loadings. The fracture quantities of QCs, i.e., field intensity factors and energy release rates, are a prerequisite. Moreover, the weight functions are applied to calculate the field intensity factors in a cracked body with both phonon and phason point forces at arbitrary locations.

INTRODUCTION

Since the discovery of quasicrystals (QC) in an Al-Mn alloy [1], the elastic theory of QCs continues to attract investigators' attention [2–8]. A 3D QC refers to a 3D solid structure with quasiperiodic arrangement in all the three directions, where phonon and phason fields exist simultaneously. Because of the intrinsic brittleness of QC materials, cracks play an important role by drastically reducing the fracture strength and fatigue limits. Recently, some experimental results have been reported which show that some 2D defects such as planar defects and cracks produced by cleavage are detected in QCs [9, 10].

The depiction of elastic problem of 3D QCs requires not only phonon fields u_i describing the mechanical displacements of the crystal system, but also phason fields w_i describing the atom arrangement along the quasiperiodic direction. Additionally, both types of fields are coupled with each other. Corresponding to the phonon and phason parameters, there are two stress fields σ_{ij} and H_{ij} associated with two strain fields ε_{ij} and w_{ij}, respectively, the latter being a new parameter in QC elasticity. According to the generalized elasticity theory of QCs [3], in the absence of body force, the following equations describe the elasticity for QCs,

$$\begin{aligned} &\varepsilon_{ij} = \left(u_{i,j} + u_{j,i}\right)/2, \quad w_{ij} = w_{i,j}, \quad \sigma_{ij,j} = 0, \quad H_{ij,j} = 0. \\ &\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ijkl}w_{kl}, \quad H_{ij} = R_{klij}\varepsilon_{kl} + K_{ijkl}w_{kl}, \end{aligned} \tag{1}$$

where i, j, k, l = 1, 2, 3, and the repeated lower case indices imply summation. C_{ijkl} and K_{ijkl} denote elastic constants in the phonon and phason fields, respectively, R_{ijkl} phonon-phason coupling elastic constants. Construct a 6D generalized displacement vector and 6×3 generalized stress tensor

$$\begin{aligned} &u_I = u_I \quad \left(I = 1,2,3\right); \quad u_I = w_{I-3} \quad \left(I = 4,5,6\right), \\ &\sigma_{Ij} = \sigma_{Ij} \quad \left(I = 1,2,3\right); \quad \sigma_{Ij} = H_{(I-3)j} \quad \left(I = 4,5,6\right). \end{aligned} \tag{2}$$

Following the steps of research progress in anisotropic elasticity [11], the general solultion of the generalized 2D problem can be expressed as

$$\boldsymbol{u} = \boldsymbol{A}\boldsymbol{f}\left(z_*\right) + \overline{\boldsymbol{A}}\overline{\boldsymbol{f}\left(z_*\right)}, \quad \boldsymbol{\sigma}_1 = -\boldsymbol{\varphi}_{,2}, \quad \boldsymbol{\sigma}_2 = \boldsymbol{\varphi}_{,1}, \quad \boldsymbol{\varphi} = \boldsymbol{B}\boldsymbol{f}\left(z_*\right) + \overline{\boldsymbol{B}}\overline{\boldsymbol{f}\left(z_*\right)}, \tag{3}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are two constant matrices, and $\boldsymbol{\varphi}$ denotes generalized stress function vector. 2D problems of QCs reduce to determining the unknown complex vector $\boldsymbol{f}$, which satisfies a given boundary condition.

A CRACK WITH UNIFORM LOADINGS APPLIED AT INFINITY

Consider a 2D problem of QCs containing a crack of length $2a$, the uniform loadings $\boldsymbol{t}_1^\infty = \left[\sigma_{11}^\infty, \sigma_{12}^\infty, \sigma_{13}^\infty, H_{11}^\infty, H_{12}^\infty, H_{13}^\infty\right]^{\mathrm{T}}$ and $\boldsymbol{t}_2^\infty = \left[\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, H_{21}^\infty, H_{22}^\infty, H_{23}^\infty\right]^{\mathrm{T}}$ are simultaneously applied at infinity. At the crack-tip $x_1 = a$, the field intensity factors can be obtained

$$\boldsymbol{K} = \left[K_{\mathrm{II}}^{\parallel}, K_{\mathrm{I}}^{\parallel}, K_{\mathrm{III}}^{\parallel}, K_{\mathrm{II}}^{\perp}, K_{\mathrm{I}}^{\perp}, K_{\mathrm{III}}^{\perp}\right]^{\mathrm{T}} = \sqrt{\pi a}\,\boldsymbol{t}_2^\infty, \tag{4}$$

where the superscripts $\parallel$ and $\perp$ associated with phonon and phason fields, respectively. The local crack tip strain energy release rate is found to be

$$G = \frac{1}{4}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{K}, \tag{5}$$

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where matrix $\boldsymbol{H}$ is defined by $\boldsymbol{H} = 2\,\mathrm{Re}\,\boldsymbol{Y}$, $\boldsymbol{Y} = \mathrm{i}\boldsymbol{A}\boldsymbol{B}^{-1}$. The total energy release rate is determined by a unique quantity in which phonon and phason fields are coupled, even if the phason stress intensity factor is zero. This shows that the phason field strongly influences the deformation and fracture of the material.

WEIGHT FUNCTIONS FOR CRACKS

The concept of crack weight functions, well known in fracture mechanics of elastic materials [12] has been extended to piezoelectric materials [13]. To calculate field intensity factors $\boldsymbol{K}^2$ due to point forces $\boldsymbol{F} = \left[P_1, P_2, P_3, Q_1, Q_2, Q_3\right]^{\mathrm{T}}$ which collocate point phonon force P_i and point phason force Q_i at any locations, the weight function method is applied:

$$\boldsymbol{K}^2 = \boldsymbol{h}\boldsymbol{F}. \tag{6}$$

For loads being applied in the vicinity of a crack tip, the weight functions are

$$\boldsymbol{h} = \frac{-1}{\sqrt{\pi a}} \boldsymbol{H}^{-1}\,\mathrm{Re}\,\boldsymbol{A} \left\langle \frac{\sqrt{z_* + a}}{\sqrt{z_* - a}} \right\rangle \boldsymbol{B}^{-1}. \tag{7}$$

For point loads applied on the right crack surfaces, where $z_\alpha = x_1$, the weight functions become

$$\boldsymbol{h} = \pm \frac{1}{2\sqrt{\pi a}} \sqrt{\frac{a + x_1}{a - x_1}} \boldsymbol{I}. \tag{8}$$

For the crack tip on the right crack surfaces, the near tip weight functions are

$$\boldsymbol{h} = \pm \frac{1}{\sqrt{2\pi r}} \boldsymbol{I}. \tag{9}$$

Each arbitrary loading configuration can be constructed by point loadings on the crack faces, following the principle of linear superposition. By integration the weight functions and prescribing loadings along the Neumann boundary condition, Eq. (7) can therefore be used to calculate stress intensity factors for distributed loadings, too.

CONCLUSIONS

The 2D problems of a crack in 3D QCs subject to far field loadings are studied. The generalized Stroh formalism is adopted here, and the explicit solutions for the coupled fields are obtained in the closed form. The crack theory of QCs, including the determination of the field intensity factors and crack tip energy release rates, is a prerequisite. The weight functions for a QC body with a crack are calculated. The weight functions provide a means of calculating the intensity factors for the crack when both phonon and phason point forces are imposed at arbitrary locations.

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