CONSERVATION LAWS FOR MATERIALS WITH A NONLINEAR POWER LAW BEHAVIOR IN ANTIPLANE DEFORMATION

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Summary Applying Noether's theorem with the help of Lie's infinitesimal invariance criterion to the functional of materials with a nonlinear power law behavior in antiplane deformation, their symmetry-transformations and the related conservation laws are obtained. For this physical nonlinear problem unlike geometrically nonlinear problem, the translation, rotation and scale change of coordinates are symmetry-transformations. Especially, the scale change of coordinates is directly related to the strain-hardening coefficient. Moreover, when the strain-hardening coefficient is close to 1, the materials turn out to be a linear material and a phenomenon of symmetry breaking occurs. And any conformal transformations in two-dimensional Euclidean space for elastic antiplane deformation are symmetry-transformations, which mean that there are countless symmetry-transformations and related conservation laws in material space. For plane elasticity, there also exists this kind of conservation laws.

DESCRIPTION OF THE PROBLEM

By using a deformation plasticity theory, the governing equations for strain-hardening elastic-plastic behaviour in antiplane deformation had been formulated (Neuber, [1]; Rice, [2]). The constitutive equations with the help of principal shear stress $\tau = (\sigma_{3i}\sigma_{3i})^{1/2}$ and strain $\gamma = (u_{3,i}u_{3,i})^{1/2}$ can be written as

$$\frac{\sigma_{3i}}{\tau_0} = (\frac{\tau}{\tau_0})^{1-n}\frac{u_{3,i}}{\gamma_0}, \quad \frac{\gamma}{\gamma_0} = (\frac{\tau}{\tau_0})^n, \tag{1}$$

where σ_{3i} and $u_{3,i}$ are the stress and strain; τ_0, γ_0 and n are the material constants and n is also called as the strain-hardening coefficient. When $n \to 1$, the first in (1) is reduced to the hooke's law in shear with the shear modulus $G = \tau_0 / \gamma_0$. In this paper, the Latin indices run from 1 to 2 for two-dimensional problem and the summation convention for repeated indices is implied. The equilibrium equation $\sigma_{3i,i} = 0$ with the help of (1) is the following

$$\gamma^2 u_{3,ii} + \frac{1-n}{n} u_{3,k}u_{3,i}u_{3,ki} = 0, \tag{2}$$

and variational problem of the strain energy density

$$W = [n\tau_0 /(1+n)\gamma_0^{1/n}]\gamma^{(1+n)/n} \tag{3}$$

with the independent displacement u_3 leads to the field equation (2) and the related nature, homogenous boundary conditions. Therefore, the requirement of Noether's theorem (Noether, [3]) is satisfied.

SYMMETRY-TRANSFORMATIONS

For the first order variational problem (3), all possible non-trivial conservation laws derived from the invariance of W can be formulated as (Olver, [4])

$$D_i(\eta\sigma_{3i} + \zeta_k S_{ik}) = 0, \quad S_{ik} = W\delta_{ik} - \sigma_{3i}u_{3,k}, \tag{4}$$

where S_{ik} is the so-called energy-momentum tensor; η and ζ_k are unknown symmetry-transformation functions of x_k and u_3 to be determined; δ_{ik} is the Kronecker delta and D_i is the total differential operator. In order to find functions η and ζ_k in the first of Eqs. (4), we expand it as follows

$$\frac{\partial\eta}{\partial x_i}u_{3,i} + [(\frac{\partial\eta}{\partial u_3} + \frac{n}{1+n}\frac{\partial\zeta_p}{\partial x_p})\delta_{ik} - \frac{\partial\zeta_k}{\partial x_i}]u_{3,k}u_{3,i} - \frac{1}{1+n}\frac{\partial\zeta_k}{\partial u_3}u_{3,k}u_{3,i}u_{3,i} = 0\cdot \tag{5}$$

Since Noether's theorem (Noether, [3]) demands that Eq. (5) vanish identically, the coefficients of all the independent linear, quadratic and cubic terms of $u_{3,k}$ must be equal to zero. This requirement gives the determining equations

$$\frac{\partial\eta}{\partial x_i} = 0, \quad 2(\frac{\partial\eta}{\partial u_3} + \frac{n}{1+n}\frac{\partial\zeta_p}{\partial x_p})\delta_{ik} - \frac{\partial\zeta_k}{\partial x_i} - \frac{\partial\zeta_i}{\partial x_k} = 0, \quad \frac{\partial\zeta_k}{\partial u_3} = 0\cdot \tag{6}$$

The elementary analysis of Eqs. (6) gives

$$\eta = Au_3 + \alpha_0, \quad \zeta_1 = \frac{1+n}{1-n}Ax_1 - \Omega x_2 + C_1, \quad \zeta_2 = \frac{1+n}{1-n}Ax_2 + \Omega x_1 + C_2, \tag{7}$$

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where α_0, A, Ω and C_i are the independent arbitrary constants. On the other hand, when $n \to 1$ for a linear material as shown in (1) – (3), the elementary analysis of Eqs. (6) gives

$$\eta = \alpha_0, \quad \zeta_1 + i\zeta_2 = \zeta(z), \tag{8}$$

where α_0 is an independent arbitrary constant and $\zeta(z)$ is an arbitrary analytic function or a conformal mapping, which has been given in Reference (Shi, [5]).

Apparently, the translation, rotation and scale change of coordinates are symmetry-transformations (7) of materials with the strain-hardening elastic-plastic behaviour in a deformation plasticity theory. It should be mentioned that there does not exist a symmetry-transformation from the scale change of coordinates for geometrically nonlinear problem (Shi, [6]). For the physical nonlinear problem in this paper, the scale change of coordinates with the constant A is a symmetry-transformation (7), which is directly related to the strain-hardening coefficient n. Moreover, for a linear elastic material with $n \to 1$ in (1) – (3), a phenomenon of symmetry breaking occurs since any conformal transformations $\zeta(z)$ in two-dimensional Euclidean space for elastic antiplane deformation are symmetry-transformations (8), which means that there are countless symmetry-transformations.

CONSERVATION LAWS

By considering the independence of arbitrary constants (7), the conservation laws follow from the first in (4) as follows,

(a) Zero position change of function u_3 $(\alpha_0 \neq 0)$: $\quad D_i\sigma_{3i} = 0$; (9)

(b) Coordinate translation $(C_k \neq 0)$: $\quad D_i S_{ik} = 0$; (10)

(c) Coordinate rotation $(\Omega \neq 0)$: $\quad D_i(x_1 S_{i2} - x_2 S_{i1}) = 0$; (11)

(d) Scale change of coordinates $(A \neq 0)$: $\quad D_i(x_1 S_{i1} + x_2 S_{i2} + \frac{1-n}{1+n}\sigma_{3i}u_3) = 0$. (12)

Clearly, expression (9) is the field equation (2) and belongs to a conservation law in physical space. Expressions (10) – (12) are called the conservation laws in material space.

As mentioned above, when the strain-hardening coefficient $n \to 1$, the materials turn out to be a linear material and expressions (1) – (6) become those for elastic antiplane deformation, which has been discussed in (Shi, [5]). If we let $u_3 = \operatorname{Im} w(z)$ with $z = x_1 + ix_2$, by using a conformal transformation $\zeta(z)$ in (8), the conservation law in integral form is given by

$$\operatorname{Im}\frac{G}{2}\oint \zeta(z)[\overline{w'(z)}]^2\, dz = 0, \tag{13}$$

which means that there are countless conservation laws in material space due to any conformal transformations $\zeta(z)$.

At last, since the real and imaginary parts of complex Kolosov-Muskhelishvili potential $\phi(z)$ satisfies also a harmonic equation in plane elasticity, we can get (Shi, [7])

$$\operatorname{Im}\frac{4}{E'}\oint \zeta(z)[\phi'(z)]^2\, dz = 0, \tag{14}$$

where $E' = E$ for plane stress, $E' = E/(1-\nu^2)$ for plane strain, E is the tensile modulus and ν Poisson's ratio. The conservation laws (13) and (14) provide the parameters to estimate the safety of sharp V-notches (Shi, [5], [7]).

CONCLUSIONS

Conservation laws for materials with a nonlinear power law behavior in antiplane deformation have been obtained, and a phenomenon of symmetry breaking is observed when the strain-hardening coefficient is close to 1. For linear material, there are countless symmetry-transformations $\zeta(z)$ in (13) and (14). Clearly, by adjusting the conformal transformation $\zeta(z)$, a finite value can be obtained from calculating the path-independent integrals (13) and (14) around a material point with any order singularity.

References

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