ON THE DISPERSION RELATION BETWEEN NONLOCAL ELASTICITY AND LATTICE DYNAMICS

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Summary This paper presents a theoretical investigation of on the dispersion relation between nonlocal elasticity and lattice dynamics. The Helmholtz type kernels for nonlocal elasticity theory are studied in 1D and 2D systems and the nonlocal elastic moduli are obtained by comparing with lattice dynamics. Different nonlocal moduli of longitudinal and transversal wave are reported.

INTRODUCTION

The classical theory of elasticity which predicts no dispersion and it is valid only for small wave numbers. Based on the nonlocal continuum theory, Lim and Wang [1] presented a complete and asymptotic representation of the 1D nonlocal nano-beam model via an exact variational principle approach In the theory of nonlocal elasticity and for a special class of kernels, the relation between nonlocal stress and the classical stress can be linked by a linear differential operator. These kernels lead to dispersion relations for plane harmonic waves, which are coincident to the well-known Born-von Kármán model of lattice dynamics [2].

The nonlocal constant e_0 of the kernel is the key issue to be determined. Eringen [2] proposed e_0 as 0.39 based on the Helmholtz type kernel and the Born-von Kármán model. He also reported $e_0 = 0.31$ by comparing with the Rayleigh surface wave and lattice dynamics [2]. Lazar et. al. [4] used a bi-Helmholtz type kernel to study the dispersion relation with $e_0 = 0.28$.

In this paper, the nonlocal continuum theory with Helmholtz type kernel is discussed. By comparing with lattice dynamics of 1D chain model and 2D cubic model, the nonlocal constant is determined.

DISPERSION RELATION BASED ON LATTICE DYNAMICS

One-dimensional model

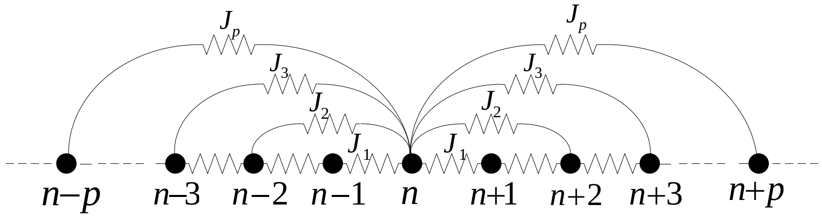

Fig. 1 Interaction forces of the nth atom in 1D chain lattice

Consider axial tension of an infinite elasticity bar, the equation of motion without body forces is

$$\rho \ddot{u} = \frac{\partial t}{\partial x} = E\int_{x-l}^{x+l} \frac{\partial \alpha\left(\left|x-x'\right|\right)}{\partial x}\frac{\partial u}{\partial x'}dx' \tag{1.1}$$

By replacing the integration terms in *Eq*. 1.1 as a summation, the 1D continuum media can be represented by an chain of atoms (Fig. 1) separated by the unit cell length a. Considering the motion of the *n*-th atom, the dispersion of the atomic chain gives

$$\omega^2/\omega_0^2 = \frac{4k^2}{a^2 J_1}\sum_{p=1}^{N} J_p \sin^2\left(\frac{kap}{2}\right) \tag{1.2}$$

where J_1 and J_p are the first and the *p*-th spring constants, respectively, and the corresponding nonlocal kernel is given by Eringen [3]

$$\omega a / c_0 = ka\left(1+e_0^2 a^2 k^2\right)^{-1/2} \tag{1.3}$$

where $\tau = e_0 a$, a the lattice parameter, $c_0 = \sqrt{E/\rho}$ the phase velocity of long wave length and e_0 the material constant including nonlocal effect. The material constant e_0 can be determined by combining *Eqs*. 1.2 and 1.3 at the

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lattice boundary $ka = \pi$. The match should be assessed by referring to the maximum and average errors in the range of $ka < \pi$ and the group velocity ($v_g = \frac{\partial \omega}{\partial k}$) at $ka = \pi$ which must vanish theoretically.

Eringen [2] compared the dispersion curve of nonlocal elasticity with the Born-von Kármán model of lattice dynamics, which led to $e_0 \approx 0.39$. The maximum error was of the order of 6% but the group velocity was baddy off at the boundary. Using the analysis presented herein, converged results considering long distant interaction can be obtained. The nonlocal constant e_0 increases from 0.386 to 2.470 and the maximum error decreases from 6.34% to 0.54% as the distant neighbourhood considered widens from 1 to 20. The group velocity approaches to zero from 0.258 for $p=1$ to 0.002 for $p=20$ and it vanishes for $p=50$ where $e_0 = 5.0$ with maximum error 0.13%.

Two-dimensional model

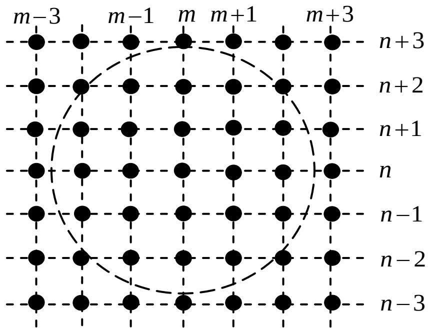

Fig. 2 Two-dimensional cubic lattice

Fig. 2 shows the crystal geometry of a simple 2D cubic lattice, where a is the cubic length. The interactions between atoms are limited in a range with radius R and the interaction potential is assumed as the Born-Mayer type. The dispersion relations are given by

$$\omega_L^2 = \frac{4}{m} \sum_{p=1}^{floor(R/a)} \sin^2\left(\frac{kap}{2}\right)\left(K_{0p} + 2 \sum_{q=1}^{floor(\sqrt{R^2-p^2a^2}/a)} K_{pq} \frac{p^2}{p^2+q^2}\right) \tag{1.4}$$

$$\omega_T^2 = \frac{8}{m} \sum_{p=1}^{floor(R/a)} \sin^2\left(\frac{kap}{2}\right) \sum_{q=1}^{floor(\sqrt{R^2-p^2a^2}/a)} K_{pq} \frac{q^2}{p^2+q^2} \tag{1.5}$$

where ω_L and ω_T, respectively, are the longitudinal and transversal wave frequencies and K_{pq} is the force constant. Based on the numerical results, the nonlocal constant e_{0L} in longitudinal direction and e_{0T} in transversal direction has similar trends with that in 1D lattice model. Distinctively, e_{0L} is always smaller than e_{0T} with increased range of nonlocal effect, which cannot be observed in 1D model. For example, $e_{0L} = 0.39, e_{0T} = 0.41$ at $a_0 = 0.2a$.

CONCLUSION

Currently, most of the lattice dynamics models which compare nonlocal elasticity are limited in 1D and short range interactions taking the nearest and the immediate neighbourhood into consideration. This paper concludes that long range interactions of atoms should be considered because the nonlocal effects influence the entire material body. The nonlocal constant increases if the long distant interactions are taken into account. On the other hand, the dispersion relation depends on not only the interaction potential but also the crystal structure of lattice model. Unlikely the 1D lattice model, the nonlocal effects have varying influences in different direction in 2D models.

References

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