STABILITY OF ANEURYSM SOLUTIONS FOR A FLUID-FILLED ELASTIC MEMBRANE TUBE

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Summary When a hyperelastic membrane tube is inflated by an internal pressure, a localized bulge will form when the pressure reaches a critical value. As inflation continues the bulge will grow until it reaches a maximum size after which it will then propagate in both directions to form a hat-like profile. The stability of such bulging solutions has recently been studied by neglecting the inertia of the inflating fluid and it was shown that such bulging solutions are unstable under pressure control. In this paper we extend this recent study by assuming that the inflation is by an inviscid fluid whose inertia we take into account in the stability analysis. This reflects more closely the situation of aneurysm formation in human arteries which motivates the current series of studies. It is shown that fluid inertia reduces the growth rate of the unstable mode and thus it has a stabilizing effect.

INTRODUCTION

The bulging phenomenon described in the Summary is well-known but its relevance to the mathematical modelling of aneurysm formation in human arteries is not appreciated until very recently. The main reason is that rubber membrane tubes and human arteries have very different constitutive behaviour and so, for instance, for the latter the inflation pressure does not exhibit a maximum. This fact has previously been used to argue that localized bulging is not mechanically possible in human arteries and so aneurysm formation can only be due to other non-mechanical factors. However, it was recently shown [1,2] that having a pressure turning point is not a necessary condition for localized bulging. It was subsequently demonstrated [3] using realistic models for human arteries that the initial formation of aneurysms can be modelled as a mechanical bifurcation phenomenon. There are two main ingredients in the analysis of [3]. Firstly, the bifurcation condition for localized bulging for a three-layer human artery does have a zero, although the corresponding critical pressure can be many orders of magnitude higher than those in the physiological range. Secondly, localized bulging is much more sensitive to localized imperfections than classical bifurcation into sinusoidal patterns [4] and as a result localized weakening of the arterial wall can easily bring the critical pressure down to the physiological range.

With the initial formation of aneurysms interpreted as a mechanical bifurcation phenomenon, there then arises the important question whether aneurysm solutions are stable or not since only stable solutions can be observed in reality, and stability will also play an important role in the subsequent evolution of an aneurysm. In [2] it was shown for a single-layer hyperelastic membrane tube that localized bulging solutions are unstable under pressure control when the fluid inertia is neglected. In this paper we investigate the effects of fluid inertia on the stability properties of localized bulging solutions.

FORMULATION

We consider an incompressible, isotropic, hyperelastic, cylindrical membrane tube that has a constant undeformed radius R and a constant undeformed thickness H. The tube is assumed to be infinitely long, and end conditions are imposed at infinity. We assume that the axisymmetry remains throughout the entire deformation so that in terms of cylindrical polar coordinates the deformation can be written as

$$r = r(Z,t), \quad \theta = \theta(Z,t), \quad z = z(Z,t), \tag{1}$$

where t denotes time. The principal directions of the deformation correspond to the lines of latitude, the meridian and the normal to the deformed surface, and the principal stretches are given by

$$\lambda_1 = \frac{r}{R}, \quad \lambda_2 = (r'^2 + z'^2)^{\frac{1}{2}}, \quad \lambda_3 = \frac{h}{H}, \tag{2}$$

where the indices 1, 2, 3 are used for the circumferential, axial and radial directions respectively, a prime represents differentiation with respect to Z, and h denotes the deformed thickness. The equations of motion take the form

$$\left[R\sigma_2 \frac{z'}{\lambda_2^2}\right]' - Prr' = \rho R \ddot{z}, \quad \left[R\sigma_2 \frac{r'}{\lambda_2^2}\right]' - \frac{\sigma_1}{\lambda_1} + Prz' = \rho R \ddot{r}, \tag{3}$$

where ρ is the density of the material, P is the internal pressure, σ_1 and σ_2 are the principal stresses in the 1- and 2-directions, respectively, and a superimposed dot denotes material time differentiation. The deformation corresponding to the initial stage of inflation is uniform and is given by $r = r_\infty,\ z = z'_\infty Z$. The uniform axial stretch z'_∞ is either specified or in the case of closed ends expressed in terms of r_∞ from constancy of the resultant axial force. In either case r_∞ can be taken as the control parameter which is related to the internal pressure P_∞. It was shown in [1,2] that for values of r_∞ in a certain interval the above equations also admit a static/standing aneurysm solution given by $r = \overline{r}(Z),\ z = \overline{z}(Z)$ that

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tends to the uniform solution in the limit $Z \to \pm\infty$. Near the bifurcation point, the aneurysm solution to leading order is governed by

$$(\overline{r}')^2 = \omega(r_\infty)(\overline{r} - r_\infty)^2 + \gamma(r_\infty)(\overline{r} - r_\infty)^3, \tag{4}$$

where the coefficients $\omega(r_\infty)$ and $\gamma(r_\infty)$ depend on the strain-energy function [5] and we note that the bifurcation value r_{cr} of r_∞ corresponds to $\omega(r_{cr}) = 0$. Thus, the weakly nonlinear aneurysm solution has the explicit expression

$$\overline{r} - r_\infty = -\frac{\epsilon\omega'(r_{cr})}{\gamma(r_{cr})}\mathrm{sech}^2\Big(\frac{\xi}{2}\Big), \quad \text{where } \epsilon = r_\infty - r_{cr}, \quad \xi = \sqrt{\epsilon\omega'(r_{cr})}Z. \tag{5}$$

When the aneurysm solution is perturbed, the fluid motion is governed by the mass balance equation and Euler's equation for an inviscid fluid:

$$\frac{\partial r}{\partial t} + v_f\frac{\partial r}{\partial z} + \frac{r}{2}\frac{\partial v_f}{\partial z} = 0, \quad \rho_f\left(\frac{\partial v_f}{\partial t} + v_f\frac{\partial v_f}{\partial z}\right) + \frac{\partial P}{\partial z} = 0, \tag{6}$$

where v_f denotes the axial fluid speed and ρ_f the fluid density divided by the undeformed wall thickness H.

STABILITY ANALYSIS

We consider axisymmetric perturbations and write

$$r = \overline{r}(Z) + w(Z)\mathrm{e}^{\eta t}, \quad z = \overline{z}(Z) + u(Z)\mathrm{e}^{\eta t}, \quad P = P_\infty + P(Z)\mathrm{e}^{\eta t}, \tag{7}$$

where η is the growth rate to be determined. If there exists one value of η whose real part is positive, then the aneurysm solution is unstable. On substituting (7) into the governing equations (3) and (6) and linearizing in terms of the perturbations $w(Z), u(Z)$ and $P(Z)$, we obtain a system of differential equations that can be written in the matrix form $\mathbf{y}' = \mathcal{M}\mathbf{y}$, where $\mathbf{y} = (u, u', w, w', P, P')^T$ and $\mathcal{M}$ is a 6×6 matrix whose components depend on the aneurysm solution.
Let $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3$ be three solutions of $\mathbf{y}' = \mathcal{M}\mathbf{y}$, and define the extended vector $\mathbf{y}^\wedge$ such that its $(i\wedge j\wedge k)$-th component is the 3×3 determinant of the i-th, j-th and k-th rows of the matrix $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3)$. We number the triplets $i\wedge j\wedge k$ such that $1 \to 1\wedge 2\wedge 3,\ \ 2 \to 1\wedge 2\wedge 4$, and etc. The vector field $\mathbf{y}^\wedge$ satisfies $\mathbf{y}^{\wedge'} = \mathcal{M}^\wedge\mathbf{y}^\wedge$, and we define the adjoint system $\mathbf{x}^{\wedge'} = -(\mathcal{M}^\wedge)^T\mathbf{x}^\wedge$. These systems are called the exterior systems. It can easily be verified that $\frac{d}{dZ}\mathbf{x}^\wedge\cdot\mathbf{y}^\wedge = 0$, and so the Evans function defined by $D(\eta) = \mathbf{x}^\wedge\cdot\mathbf{y}^\wedge$ is independent of Z. For η in the right half complex-plane, the matrix $\mathcal{M}_\infty$ (i.e. $\mathcal{M}$ in the limit $Z\to\infty$) has three eigenvalues $k_1(\eta), k_2(\eta), k_3(\eta)$ in the left half-plane. Thus the asymptotic matrix $\mathcal{M}^\wedge_\infty$ has simple left-most eigenvalue $k^\wedge(\eta) = k_1(\eta) + k_2(\eta) + k_3(\eta)$ for η in the right half-plane. There are then solutions $\mathbf{y}^\wedge$ and $\mathbf{x}^\wedge$ of the exterior systems such that

$$\lim_{Z\to\infty}\mathrm{e}^{-k^\wedge(\eta)Z}\mathbf{y}^\wedge(\eta, Z) = \mathbf{r}^\wedge(\eta), \quad \lim_{Z\to-\infty}\mathrm{e}^{k^\wedge(\eta)Z}\mathbf{x}^\wedge(\eta, Z) = \mathbf{l}^\wedge(\eta), \tag{8}$$

where $\mathbf{r}^\wedge(\eta)$ and $\mathbf{l}^\wedge(\eta)$ are the eigenvectors associated with $k^\wedge(\eta)$. Moreover, because the eigenvalue is simple, the constructions of Alexander and Sachs [6] are valid and so $D(\eta)$ is analytic in the entire right half complex-plane of η and is real for real η. The growth rate is then determined from $D(\eta) = 0$ numerically.
Near the bifurcation point where $r_{cr} - r_\infty = -\epsilon \ll 1$, we have $w = (-2\epsilon\omega'(r_{cr})/3\gamma(r_{cr}))\cdot V(\xi, \tau)$ and

$$\frac{\partial^2 V}{\partial\xi^2} - c_{1n}\frac{\partial^2 V}{\partial\tau^2} = c_2\frac{\partial^4 V}{\partial\xi^4} + c_3\frac{\partial^2 V^2}{\partial\xi^2}, \tag{9}$$

where c_{1n}, c_2, c_3 are known constants and $\tau = \epsilon t$. The constant c_{1n} reduces to the c_1 in [2] when fluid inertia is neglected. The growth rate for the only unstable mode is then given explicitly by $\eta = (c_1/c_{1n})\eta_0$, where $\eta_0 = \epsilon\sqrt{3/16c_1}$ is the growth rate found in [2] when the fluid inertia is neglected. We find that c_{1n} is always greater than c_1 when $\rho_f \neq 0$. Thus, fluid inertia reduces the growth rate and therefore it has a stabilizing effect in the weakly nonlinear limit.
We shall present numerical results that show that fluid inertia has a stabilizing effect over the entire fully nonlinear regime.

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