CONSISTENT BEAM THEORIES

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Summary A Timoshenko-type beam theory is developed without the use of any a-priori assumptions or correction factors directly from the three-dimensional theory of linear elasticity by the consistent-modeling approach. The approach proceeds in three steps. First an exact one-dimensional representation is derived from the three-dimensional problem by the use of a variational principle and series expansions. The exact problem is given by infinite power-series in parameters characterizing the relative thickness of the beam in cross-section measures. Secondly, these series are truncate after certain powers, which defines the approximation order of the resulting theory. Finally, the resulting ODE systems are pseudo-reduced to single ODEs.

The first-order approximation coincides with the Euler-Bernoulli beam theory. The second-order approximation yields a yet unknown Timoshenko-type theory, which will be used to assess and validate existing theories.

THE AIM

Several papers have been published concerning the shear-correction factor of Timoshenko's beam theory (see [1], [2]) without reaching an overall consent (compare [3]). We seek a Timoshenko-like theory directly derivable from the equations of three-dimensional linear elasticity by the method of consistent approximation, avoiding any use of a-priori assumptions or correction factors.

THE APPROACH

We basically follow the same consistent-modeling approach introduced in [4] and [5] for isotropic plates and extended in [6] and [8] for anisotropic plates. Here the problem of a homogeneous isotropic (E, ν) three-dimensional beam $(b, h \ll l)$, which is loaded by a length-coordinate dependent boundary-traction q on the top-side (see figure 1), in the setting of linear elasticity is investigated. The original coordinate system shall be x, y, z, oriented at the center of one side of the beam, whereas $\xi := x/l, \eta := y/l, \zeta := z/l$ denote dimensionless coordinates.

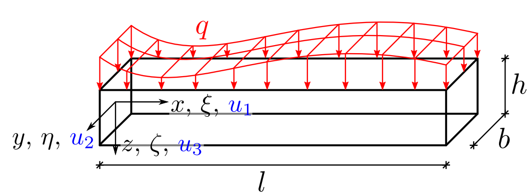

Figure 1. geometry and load case

The overall elastic potential of a random displacement field φ, which is not necessarily fulfilling the prescribed boundary conditions u^* at some part of the lateral surface areas $\partial\Omega_0$ of the beam Ω, is (in the absence of volume forces) given by

$$\Pi(\varphi_i) = \int\limits_{\Omega} \frac{1}{2} E_{ijkl}\varphi_{k,l}\varphi_{i,j}\, dV - \int\limits_{A^-} q_i\varphi_i\, dA - \int\limits_{\partial\Omega_0} E_{ijkl}\varphi_{k,l}n_j\left(\varphi_i - u_i^*\right) dA,$$

where E denotes the forth-order elasticity tensor, n is the outer-normal-unit vector and A^- is the top area of the beam. By Taylor-series expansions every displacement field φ can be written in the form

$$\varphi_i(\xi, \eta, \zeta) = l \sum_{n=0}^{\infty} \sum_{m=0}^{n} {}^{m;n-m}\varphi_i(\xi)\zeta^m\eta^{n-m}. \tag{1}$$

Following the standard approach of variation theory, we assume that φ is the sum of the equilibrium solution u and a variation εv and thus ${}^{mp}\varphi_i = {}^{mp}u_i + \varepsilon\, {}^{mp}v_i$. The necessary condition for an equilibrium solution is, that the first variation of the elastic potential vanishes $\delta\Pi = \frac{\partial\Pi}{\partial\epsilon}\big|_{\epsilon=0} = 0$. By applying the variational lemma for each coefficient ${}^{mp}v_i$ occurring in this condition, we find an infinite system of countably many ODEs in stress-resultants ${}^{mp}m_{ij}$ and load-resultants ${}^{mp}p_i$ and corresponding boundary conditions

$$-\frac{1}{l}\, {}^{mp}m'_{i1}(\xi) + p \cdot {}^{m(p-1)}m_{i2}(\xi) + m \cdot {}^{(m-1)p}m_{i3}(\xi) - {}^{mp}p_i(\xi) = 0, \quad \xi \in (0,1), \quad m, p \in \mathbb{N}_0, \tag{2}$$

$${}^{mp}m_{j1}(\xi) - {}^{mp}m^*_{j1}(\xi) = 0 \quad \text{or} \quad {}^{mp}u_i(\xi) - {}^{mp}u^*_i(\xi) = 0, \quad \xi \in \{0,1\}, \quad m, p \in \mathbb{N}_0, \tag{3}$$

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where

$$
\begin{aligned}
{}^{mp}m_{ij}(\xi) \quad &:= \quad l^{(m+p+2)} \int\limits_{-\frac{h}{2l}}^{\frac{h}{2l}} \int\limits_{-\frac{b}{2l}}^{\frac{b}{2l}} \sigma_{ij}(\xi,\eta,\zeta)\, \zeta^m\, \eta^p\, d\eta d\zeta,\ m,p \in \mathbb{N}_0, \qquad (4)\\
{}^{mp}p_3(\xi) \quad &:= \quad l^{m+p+1} \frac{2}{p+1} \left(\frac{h}{2l}\right)^m \left(\frac{b}{2l}\right)^{p+1} q(\xi,\eta=0),\ m \in \mathbb{N}_0,\ p \in \{0,2,4,...\}\,.
\end{aligned}
$$

The problem (2) and (3) is indeed equivalent to the three-dimensional problem (compare [6]) and therefore an exact one-dimensional representation of the three-dimensional problem of elasticity.

In the linear theory we have $\sigma_{ij} = E_{ijkl}u_{k,l}$. By inserting this equation and the series expansion (1) into the stress-resultants (4), we can rewrite the field-equations (2) and stress-boundary conditions (3) completely in terms of displacement coefficients. Each equation of (2) becomes a power-series in characteristic parameters $c^2 := \frac{h^2}{12l^2}$ and $d^2 := \frac{b^2}{12l^2}$, since

$$
\int\limits_{-\frac{h}{2l}}^{\frac{h}{2l}} \int\limits_{-\frac{b}{2l}}^{\frac{b}{2l}} \zeta^q \eta^r d\eta d\zeta = \frac{bh}{l^2}\frac{3^{q/2}}{q+1}\frac{3^{r/2}}{r+1}c^q d^r \text{ for } q,r \text{ even and } 0 \text{ otherwise.}
$$

Furthermore, each equation contains countably many unknown displacement coefficients u_i^l, so that the exact representation (2) and (3) is not handable in practice. Since c and d describing the relative thickness of the beam and are therefore assumed to be very small $(b, h \ll l)$, we truncate the power-series leading to approximative representations. Precisely we neglect every term containing a factor $c^{2q}d^{2r}$ with $2q + 2r > 2n$ for the n-th-order approximation, which leads always to a finite ODE system in finitely many unknown displacement coefficients and a matching number of corresponding boundary conditions.

By a certain kind of pseudo-reduction algorithm (see [7]), which respects the fact that the given ODEs are truncated power-series, it is possible to reduce the ODE system to a single ODE, which is to be solved.

THE RESULTS

The zeroth-order approximation only allows rigid-body motions as solutions.

The first-order approximation turns out to be exactly the Euler-Bernoulli theory, with the displacement of the neutral axis ${}^{00}w$ as independent variable and the two classical boundary conditions

$$
\begin{aligned}
EI\,{}^{00}w^{IV}(\xi) = l^3 b\, q(\xi,\eta=0), \quad \xi \in (0,1),\\
\left(Q :=^{00} m_{13} =^{00} m_{13}^* \text{ or } {}^{00}w =^{00} w^*\right) \text{ and } \left(M :=^{10} m_{11} =^{10} m_{11}^* \text{ or } {}^{00}w' =^{00} w'^*\right), \quad \xi \in \{0,1\}\,.
\end{aligned}
$$

The second-order approximation is a Timoshenko-type theory, which is yet unpublished. It is formulated in an averaged cross-section displacement $\tilde{w}$ (also containing y-direction displacements)

$$
\begin{aligned}
EI\,\tilde{w}^{IV}(\xi) = l^3 b\left[q(\xi,\eta=0) - \frac{6}{5}(2+\nu)\,c^2\,q''(\xi,\eta=0)\right], \quad \xi \in (0,1),\\
\left(Q :=^{00} m_{13} =^{00} m_{13}^* \text{ or } \tilde{w} = \tilde{w}^*\right) \text{ and } \left(M :=^{10} m_{11} =^{10} m_{11}^* \text{ or } \psi = \psi^*\right), \quad \xi \in \{0,1\}\,.
\end{aligned}
$$

The variable ψ is dependent on $\tilde{w}$ and is the second-order approximation of the averaged cross-section deflection.

In the talk, the consistent modeled theory will be uses to assess and validate various existing beam theories.

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