IDENTIFICATION OF SMALL MULTIPLE DEFECTS IN 3D LINEAR ELASTIC BODY

Efim I. Shifrin[a)] & Pavel S. Shushpannikov
A.Yu. Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences,
Prosp. Vernadskogo 101-1, Moscow, 119526, Russia

Summary A problem of identification of a finite number of small, well-separated defects (cavities, rigid or linear elastic inclusions) in an isotropic, linear elastic 3d body, using results of one static test, is considered. It is assumed that the loads and displacements are measured on the external boundary of the body. A method is developed for determination of the centers of defects projections on an arbitrary plane. If the defects are spherical cavities or rigid inclusions their radii can be determined also. Numerical examples are considered.

INTRODUCTION

The reciprocity gap functional (RGF) method was successfully used for solving elastostatic inverse problems in [1-4]. The solved problems concerned mainly the cases of identification of a single defect. Below we develop a method, based on the RGF, for identification of a finite number of defects using results of one static test. The problem is reduced to the problem of identification of the projections of the defects on an arbitrary plane due to the use of the regular elastic fields of specific type. The obtained 2d problem is turned out similar to the problem of identifying simple poles of a meromorphic function. It enabled to use in the considered problem the known methods.

STATEMENT OF THE PROBLEM

Let $V \subset R^3$ be a bounded simply connected domain with a boundary ∂V. $G_k \subset V$, $k = 1, 2, \cdots, n$ are small, simply connected subdomains. We suppose that $\bar{G}_i \cap \bar{G}_j = \varnothing$, $i \neq j$, where $\bar{G}_k$ is a closure of the subdomain G_k. Let us suppose that an isotropic linear elastic body with a shear modulus μ_M and Poisson ratio ν_M occupies the domain $\Omega = V \setminus \bar{G}$, where $\bar{G} = \bigcup_{k=1}^{n} \bar{G}_k \subset V$. The defects G_k can be cavities or inclusions (rigid or linear elastic). If G_k is a cavity we suppose that its boundary ∂G_k is unloaded. If G_k is an inclusion, it is supposed complete bonding between the matrix and inclusion. Let us introduce Cartesian coordinates $Ox_1x_2x_3$. We suppose that the loads $\mathbf{t}^d = \left(t_1^d, t_2^d, t_3^d\right)$ are applied to the boundary ∂V and displacements $\mathbf{u}^d = \left(u_1^d, u_2^d, u_3^d\right)$ are measured on ∂V. We will mark with a superscript d the stress-strain state in the body Ω: σ_{ij}^d is the stress tensor, e_{ij}^d is the strain tensor, $\mathbf{u}^d$ is the displacement vector, $t_i^d = \sigma_{ij}^d n_j$, where $n = \left(n_1, n_2, n_3\right)$ is a unit outward normal to the boundary ∂V. Below we will suppose that the defects are linear elastic inclusions. The cases of cavities and rigid inclusions can be considered as limit cases when the elastic moduli tend to zero or infinity, respectively. The stress-strain state in the inclusion G_k we mark with the superscript Ik ($\sigma_{ij}^{Ik}, e_{ij}^{Ik}, \mathbf{u}^{Ik} = \left(u_1^{Ik}, u_2^{Ik}, u_3^{Ik}\right)$). The shear modulus and Poisson ratio of the inclusion G_k we denote by μ_k and ν_k, respectively. According to our suppositions the following equations are valid for $x = \left(x_1, x_2, x_3\right) \in \Omega$:

$$e_{ij}^d = \frac{1}{2}\left(u_{i,j}^d + u_{j,i}^d\right),\ \left(i = 1,2,3; j = 1,2,3\right);\ \sigma_{ij}^d = 2\mu_M\left(\frac{\nu_M}{1-2\nu_M}\theta^d\delta_{ij} + e_{ij}^d\right),\ \theta^d = \sum_{k=1}^{3} e_{kk}^d;\ \sigma_{ij,j}^d = 0 \quad (1)$$

The elastic field with the superscript Ik satisfies in the domain G_k the equations, analogical to equations (1) with the replacement of the values μ_M and ν_M by the values μ_k and ν_k, respectively. We will call the elastic fields in the body V without defects as regular elastic fields and mark by a superscript r ($\sigma_{ij}^r, e_{ij}^r, \mathbf{u}^r$). The RGF, depending on two stress states with superscripts d and r, is defined as follows:

$$RG\left(d, r\right) = \int_{\partial V} \left(t_i^d u_i^r - t_i^r u_i^d\right) dS, \quad t_i^r = \sigma_{ij}^r n_j \quad (2)$$

[a)] Corresponding author. Email: shifrin@ipmnet.ru

The problem is to reconstruct the defects G_k using the known loads $\mathbf{t}^d$ and displacements $\mathbf{u}^d$ on the boundary ∂V. Because the vector-functions $\mathbf{t}^d$ and $\mathbf{u}^d$ are known on ∂V, the values $RG(d,r)$ can be calculated for all regular fields r. So, the problem will be solved if we express the parameters of the domains G_k by means of the values $RG(d,r)$. According to the results [2], expression (2) can be rewritten in the following form:

$$RG(d,r) = \sum_{k=1}^{n} \int_{G_k} \Delta\sigma_{ij}^{Ik} e_{ij}^{r}\, dx \tag{3}$$

where $\Delta\sigma_{ij}^{Ik} = \sigma_{ij}^{Ik} - \bar{\sigma}_{ij}^{Ik}$, $\bar{\sigma}_{ij}^{Ik}$ are the stresses corresponding to the strains e_{ij}^{Ik} in the material with the shear modulus μ_M and Poisson ratio ν_M.

REDUCTION OF THE PROBLEM TO THE PROBLEM OF THE DEFECTS PROJECTIONS IDENTIFICATION

Let us denote the centers of the defects G_k by $x^k = \left(x_1^k, x_2^k, x_3^k\right)$ and the volumes of the domains G_k by $|G_k|$. We suppose that the diameters of the defects are small and the defects are well-separated. It follows from our suppositions that in this case equation (3) can be rewritten in the form:

$$RG(d,r) \approx \sum_{k=1}^{n} \Delta\sigma_{ij}^{Ik}\left(x^k\right) e_{ij}^{r}\left(x^k\right) |G_k| \tag{4}$$

Consider, for example, projections of the defects on the plane x_1x_2. Let $f(z) = u(x_1,x_2) + i\mathrm{v}(x_1,x_2)$, $z = x_1 + ix_2$ be a holomorphic function. It follows from the Cauchy-Riemann equations that the vectors of displacements $\mathbf{u}^r = (u, -\mathrm{v}, 0)$ and $\mathbf{u}^\rho = (-\mathrm{v}, -u, 0)$ satisfy the Lame equations (1). The strain tensors corresponding to the fields of displacements $\mathbf{u}^r$ and $\mathbf{u}^\rho$ are as follows:

$$e_{11}^r = u_{,1};\ e_{22}^r = -e_{11}^r;\ e_{12}^r = -\mathrm{v}_{,1};\ e_{13}^r = e_{23}^r = e_{33}^r = 0;\ e_{11}^\rho = -\mathrm{v}_{,1};\ e_{22}^\rho = -e_{11}^\rho;\ e_{12}^\rho = -u_{,1};\ e_{13}^\rho = e_{23}^\rho = e_{33}^\rho = 0 \tag{5}$$

It follows from the equations (4), (5)

$$RG(d,r) - iRG(d,\rho) \approx \sum_{k=1}^{n} \left[\left(\Delta\sigma_{11}^{Ik}\left(x^k\right) - \Delta\sigma_{22}^{Ik}\left(x^k\right)\right) + 2i\Delta\sigma_{12}^{Ik}\left(x^k\right)\right] f'(z_k) |G_k| \tag{6}$$

Let us take, for example, $f(z) = f_m(z) = \dfrac{z^{m+1}}{(m+1)L^m}$, $m = 0,1,2,\cdots$, where L has the dimension of length. In this case $f_m'(z) = w^m$, $w = z/L$, and equations (6) have the following form:

$$\sum_{k=1}^{n} a_k w_k^m = b_m \tag{7}$$

Equations (7) are appeared in the problem of identification of simple poles of a meromorphic function. There are several approaches for solving such problem. We used an approach proposed in [5]. Using this approach it is possible to determine the centers $w_k = z_k / L$ of the defects projections on the plane x_1x_2 and the values $a_k = \left[\left(\Delta\sigma_{11}^{Ik}\left(x^k\right) - \Delta\sigma_{22}^{Ik}\left(x^k\right)\right) + 2i\Delta\sigma_{12}^{Ik}\left(x^k\right)\right]|G_k|$. If we suppose that the defects are spherical cavities or spherical rigid inclusions, then it is possible to determine their radii using the values a_k. Considered numerical examples confirmed efficiency of the proposed identification method.

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References

[1] Andrieux S., Ben Abda A., Bui H. Reciprocity principle and crack identification. *Inverse Problems* **15** 59-65, 1999.

[2] Shifrin E.I., Shushpannikov P.S. Identification of a spheroidal defect in an elastic solid using a reciprocity gap functional. *Inverse Problems* **26** 055001, 2010.

[3] Shifrin E.I. Ellipsoidal defect identification in an elastic body from the results of a uniaxial tension (compression) test. *Mechanics of Solids* **45** 417-426, 2010.

[4] Shifrin E.I., Shushpannikov P.S. Identification of an ellipsoidal defect in an elastic solid using boundary measurements. *International Journal of Solids and Structures* **48** 1154-1163, 2011.

[5] Kang H., Lee H. Identification of simple poles via boundary measurements and application of EIT. *Inverse Problems* **20** 1853-1863, 2004.