ANALYTICAL SOLUTION OF TRANSVERSE CRACK PROBLEM UNDER A THIN LAYER

Boris Smetanin*, Boris Sobol**, Alexander Krasnoschekov**

*Department of Theoretical Continuum Mechanics, South Federal University, Rostov-on-Don, 344006, Russia

**Department of Information Technologies, Don State Technical University, Rostov-on-Don, 344000, Russia

Summary We consider the plane static problem of elasticity theory for a half-plane weakened by a straight transverse crack. The boundary of the half-plane is reinforced by a thin flexible coating. The crack is maintained in the opened state by normal forces applied to its coasts. Application of generalized Fourier transform reduces the problem to a solution of singular integral equation of first kind with respect to the derivative of crack opening. The solution of resulting integral equation was built by collocation and small parameter methods. The values of stress intensity factor near the edges of the crack were found.
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PROBLEM STATEMENT

Materials with various functional coatings are widely used in modern industry. Thermal spraying and other technologies allow to create various coatings of arbitrary thickness, adequately protecting the base material from environmental influence. But while the base material is insulated from outside effects, its mechanical properties remain the same due to small thickness of coating. This means that materials with functional coatings are subject to internal flaws of various kinds.
For this purpose a problem of equilibrium of an elastic half-plane $y \le 0$ weakened by a straight perpendicular to the boundary crack of length $2a$ (Fig. 1) is considered. Center of the crack is located at the distance h from the surface. The boundary of half-plane is reinforced by the coating. Distributed normal forces of intensity $\sigma_x = p = const$ are maintaining the crack in the opened state.

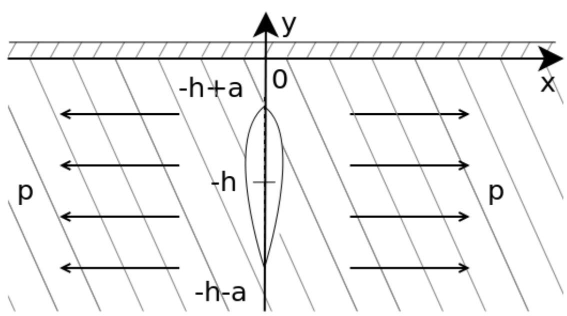

Figure 1. Half-plane weakened by a straight perpendicular to the boundary crack.

The problem is described by equations of equilibrium of displacements under the following half-plane boundary conditions at $y = 0$ [1]:

$$\begin{cases} 4G_1 h_1 u^{''} = (1-\nu_1)\tau_{xy} + 2\nu_1 h_1 \sigma_y^{'}; \\ \sigma_y = 0; \end{cases} \tag{1}$$

G_i, ν_i - shear modulus and Poisson's ratio, for the layer ($i = 1$) and the half-plane ($i = 2$), h_1 - layer thickness, τ_{xy}, σ_y- stress tensor components in the half-plane. Derivatives are taken respectively to *x*.
For discontinuities of the components of displacement vector *u* and *v* at the crack axis we introduce the following notation:

$$u|_{-0}^{+0} = X(y), \left.\frac{\partial v}{\partial x}\right|_{-0}^{+0} = \Psi(y) \tag{2}$$

Symmetry of the problem relative to *x= 0* and absence of shear stresses at the surfaces of the crack allow to connect the aforementioned functions: $\Psi(y) = -X^{'}(y)$. Further we apply generalized integral Fourier transform with respect to *x* to the equilibrium equations in displacements, considering (2) and (3). We solve these equations and find functions *u* and *v* in Fourier domain:

$$\begin{cases} U = -\frac{isgn(\alpha)}{4(1-\nu)} \int_{-h+a}^{-h-a} X(\eta)(3-2\nu-|\alpha||\eta-y|)e^{-|\alpha||\eta-y|}d\eta + |\alpha|ye^{|\alpha|y}C_1 + e^{|\alpha|y}C_2; \\ V = -\frac{1}{4(1-\nu)} \int_{-h+a}^{-h-a} X(\eta)sgn(y-\eta)(2\nu-|\alpha||\eta-y|)e^{-|\alpha||\eta-y|}d\eta + i(|\alpha|y-\kappa)C_1 + isgn(\alpha)e^{|\alpha|y}C_2; \end{cases} \tag{3}$$

Then we find C_1 and C_2 from boundary conditions (1), and afterwards apply inverse Fourier transform. As a result, the

a) Boris Sobol. E-mail: b.sobol@mail.ru.

problem is reduced to the solution of singular integral equation of the I-st kind:

$$\begin{gathered}\int_{-h-a}^{-h+a} X'(\eta)\left[\frac{1}{\eta-y}+S(\eta,y)\right]d\eta=-\frac{2\pi p}{\theta_2};S(\eta,y)=\frac{y^2-\eta^2+4y\eta}{(n+y)^3}-\frac{12(a\varepsilon-\eta)(2a\varepsilon-y)E(\eta,y)+4a\varepsilon\eta-8a^2\varepsilon^2}{(n+y)^3}\\ x=0;-h-a<y<-h+a;\theta_i=\frac{G_i}{(1-\nu_i)};\varepsilon=\frac{4h_1}{a}\frac{\theta_2}{\theta_1};E(\eta,y)=e^{-\frac{1}{a\varepsilon}(\eta+y)}\int_1^\infty\frac{e^{\frac{1}{a\varepsilon}(\eta+y)t}}{t^4}dt\end{gathered} \tag{4}$$

The derivative of crack opening $X'(\eta)$ has to be determined here. The singular part of the kernel of integral equation (4) corresponds to the problem of a crack in an infinite elastic plane [2]. When $\varepsilon\to 0$ the regular part corresponds to known case of the problem for half-plane with a transverse crack and free boundary [3, 4].

TREATMENT

After transition to dimensionless variables $|\zeta|<1,|z|<1$ instead of η and y the solution of the integral equation is built by collocation method in the form which takes into account the singularity in vicinity of the crack tip:

$$g'(\zeta)=\frac{1}{\sqrt{1-\zeta^2}}\sum_{n=0}^{N}X_nT_n(\zeta) \tag{5}$$

where T_n - Chebyshev polynomials of the I type.
Thus we need to find X_n from a system of linear algebraic equations with coefficients a_{ij}:

$$a_{ij}=\int_{-1}^{1}T_j(\zeta)\frac{1}{\sqrt{1-\zeta^2}}[\frac{1}{(\zeta-z_i)}+K(\zeta,z_i)]d\zeta \tag{6}$$

The singular part of (6) is known integral [5] and regular part can be calculated numerically.
Also we can asymptotically estimate this solution with well known small parameter method. First we need to expand the kernel of integral equation (4) into Maclaurin series on $\lambda=\frac{a}{h}<1$: $K(\zeta,z)=\sum_{i=1}^{N}k_i\lambda^i+O(\lambda^{N+1})$.
Then, consequentially solve following chain of integral equations:

$$\int_{-1}^{1}\frac{1}{t-z}g_0'dt=\pi p;\int_{-1}^{1}\left[\frac{1}{t-z}g_1'+k_1g_0'\right]dt=0;\ldots;\int_{-1}^{1}\left[\frac{1}{t-z}g_n'+\sum_{m=0}^{n-1}k_{n-m}g_m'\right]dt=0,n=1,2,... \tag{7}$$

So that $g'(z)=\sum_{i=1}^{N}g_i'(z)\lambda^i+O(\lambda^{N+1})=g_\infty'(z)N(z);g_\infty'(z)=\frac{-pz}{\sqrt{1-z^2}}$ - solution for a crack in an infinite plane [2],

$$N(z)=1+\frac{\pi}{4}\lambda^2+\frac{\pi}{4}(z-\varepsilon)\lambda^3+\frac{\pi}{128}\left(12(z-2\varepsilon)^2+8\pi+3\right)\lambda^4+O(\lambda^5) \tag{8}$$

$N(z)$ - influence factor [6], reflects an influence of various geometrical and physical parameters on the solution.

RESULTS AND CONCLUSIONS

1. The considered problem was reduced to a singular integral equation of the first kind.
2. The solution of the integral equation was obtained by both small parameter and collocation methods.
3. The obtained solution, particularly allow us to calculate the intensity of normal stresses in the neighborhood of closest to the surface (most dangerous) crack tip: $K_I=-\lim_{z\to 1-0}\theta_2\sqrt{2\pi(1-z)}g'(z)$
4. It should be noted that in (8) the influence of the free surface starts from λ^2 multiplier and is positive, while the influence of coating starts only from λ^3 and is negative which means that stress reduction caused by coating is an order less than stress magnification caused by a free surface. Direct calculations by collocation method confirm this fact. So functional coatings along with protective functions also reduce the stress intensity in the crack vicinity.

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