A CONSTITUTIVE MODEL FOR A LINEARLY ELASTIC PERIDYNAMIC BODY

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Summary The displacement state, rather than the extension scalar state, is used to define infinitesimal strains in the state-based peridynamic theory. A strain energy function of a linearly elastic peridynamic material is then proposed, which is used to obtain the force vector state and the associated modulus state for this material. In the limit of small horizon, the peridynamic coefficients that appear in the strain energy are determined in terms of three elastic coefficients.

INTRODUCTION

The peridynamic theory is an extension of the classical continuum theory in which a material point interacts directly with other material points separated from it by a finite distance. The main strength of the theory is to apply its basic equations directly on either a surface of discontinuity, such as a crack, or, a continuum in which discontinuities may appear as a result of deformation. Away from these places, the deformation is smooth, causing the peridynamic theory to yield the same governing equations in the limit of vanishing distances between material points. Silling et al. [2] introduced the state-based model of the peridynamic theory, which permits the response of a material at a point to depend collectively on the deformation of all bonds connected to this point. Recently, Silling [1] investigated the response of a state-based peridynamic material for a small deformation superposed on a large deformation and introduced the modulus state, which expresses the material properties of the linearized material response. If the material is elastic, the modulus state is obtained from the second derivative of the strain energy function. This modulus state is analogous to the fourth-order elasticity tensor in the classical continuum theory. In the case that the deformation of a body is classically smooth, Silling [1] derived expressions that relate the modulus state to the elasticity tensor of the classical theory. In addition, both Silling et al. [2] and Silling [1] proposed strain energy functions for isotropic materials that depend on the extension scalar state, which is the change in length between two particles due to deformation. In this work we propose a strain energy function for a linearly elastic peridynamic material that depends on the relative displacement field between particles through measures of strain that are analogous to the measures of strain defined in the classical linear theory. The strain energy function is quadratic in these measures and depends on four elastic peridynamic coefficients. Using the expressions derived by Silling [1], we obtain expressions for three of these coefficients in terms of three elastic constants from the classical linear theory. The fourth peridynamic coefficient is left undetermined.

A LINEAR PERIDYNAMIC THEORY

Let $\mathcal{B} \subset \mathbb{E}^3$ be the undistorted reference configuration of an elastic solid body at time $t = 0$ and let $\boldsymbol{\chi}(\mathbf{x}, t)$ be the position of the particle $\mathbf{x} \in \mathcal{B}$ at time $t \geq 0$. Constitutive modeling within the peridynamic theory considers the collective deformation at each time t of all the material within a neighborhood of an arbitrary point $\mathbf{x}_0 \in \mathcal{B}$. We shall omit writing the time variable t. Here, this neighborhood is a sphere of radius δ, called the *horizon*, centered at $\mathbf{x}_0$, which we denote as $\mathcal{H}_\delta(\mathbf{x}_0) \subset \mathcal{B}$. For any $\mathbf{x} \in \mathcal{B}$ such that $|\mathbf{x}-\mathbf{x}_0| < \delta$, the vector $\boldsymbol{\xi} := \mathbf{x}-\mathbf{x}_0$ is called a bond to $\mathbf{x}_0$. We define the difference deformation state at $\mathbf{x}_0$ as $\underline{\boldsymbol{\chi}}\langle\boldsymbol{\xi}\rangle := \boldsymbol{\chi}(\mathbf{x})-\boldsymbol{\chi}(\mathbf{x}_0)$. The displacement field $\mathbf{u}(\mathbf{x})$ associated with the motion $\chi(\mathbf{x})$ at $\mathbf{x} \in \mathcal{B}$ is defined by $\mathbf{u}(\mathbf{x}) := \boldsymbol{\chi}(\mathbf{x}) - \mathbf{x}$, and the corresponding difference displacement state at $\mathbf{x}_0 \in \mathcal{B}$ is defined by $\underline{\mathbf{u}}\langle\boldsymbol{\xi}\rangle := \mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0)$.

The strain energy function in the standard peridynamic theory defines the value of the free energy at each

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$\mathbf{x}_0 \in \mathcal{B}$ in terms of the deformation state at $\mathbf{x}_0$ in the horizon $\mathcal{H}_\delta(\mathbf{x}_0)$. Here, we take the strain energy as

$$W(\mathbf{x}_0)[\underline{\mathbf{u}}] = \int_{\mathcal{H}_\delta(\mathbf{x}_0)} \int_{\mathcal{H}_\delta(\mathbf{x}_0)} \omega(|\boldsymbol{\xi}|, |\boldsymbol{\eta}|) \Big[\frac{\alpha_1}{2} \big(\varepsilon[\underline{\mathbf{u}}](\boldsymbol{\xi})\big)^2 + \alpha_2\, \varepsilon[\underline{\mathbf{u}}](\boldsymbol{\xi})\, \varepsilon[\underline{\mathbf{u}}](\boldsymbol{\eta}) + \frac{\alpha_3}{2} \big(\gamma[\underline{\mathbf{u}}](\boldsymbol{\xi}, \boldsymbol{\eta})\big)^2 + \alpha_4\, \gamma[\underline{\mathbf{u}}](\boldsymbol{\xi}, \boldsymbol{\eta})\, \varepsilon[\underline{\mathbf{u}}](\boldsymbol{\xi}) \Big]\, dv_{\boldsymbol{\eta}}\, dv_{\boldsymbol{\xi}}\,, \tag{0.1}$$

where $\omega(\cdot, \cdot)$ is a given weighting function satisfying $\omega(|\boldsymbol{\xi}|, |\boldsymbol{\eta}|) = \omega(|\boldsymbol{\eta}|, |\boldsymbol{\xi}|)$, $\alpha_i,\ i = 1, \ldots, 4$, are peridynamic constants to be determined below, $\varepsilon[\underline{\mathbf{u}}](\boldsymbol{\xi}) := \mathbf{e}(\boldsymbol{\xi}) \cdot \underline{\mathbf{u}}\langle\boldsymbol{\xi}\rangle/|\boldsymbol{\xi}|$ is an infinitesimal normal strain state with $\mathbf{e}(\boldsymbol{\xi}) := \boldsymbol{\xi}/|\boldsymbol{\xi}|$, and $\gamma[\underline{\mathbf{u}}](\boldsymbol{\xi}, \boldsymbol{\eta}) = \Big(\mathbf{e}(\boldsymbol{\xi}, \boldsymbol{\eta}) \cdot \underline{\mathbf{u}}\langle\boldsymbol{\xi}\rangle/|\boldsymbol{\xi}| + \mathbf{e}(\boldsymbol{\eta}, \boldsymbol{\xi}) \cdot \underline{\mathbf{u}}\langle\boldsymbol{\eta}\rangle/|\boldsymbol{\eta}|\Big)\Big/2\,,\ \ |\boldsymbol{\xi}| < \delta\,,\ |\boldsymbol{\eta}| < \delta$, is an infinitesimal shear strain state, with $\mathbf{e}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \big(\mathbf{1} - \mathbf{e}(\boldsymbol{\xi}) \otimes \mathbf{e}(\boldsymbol{\xi})\big)\mathbf{e}(\boldsymbol{\eta})/\sin\alpha = \big(\mathbf{e}(\boldsymbol{\eta}) - \mathbf{e}(\boldsymbol{\xi})\cos\alpha\big)/\sin\alpha$ and α being the smallest included angle between $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

The corresponding vector force state at $\mathbf{x}_0$, which for fixed difference displacement state $\underline{\mathbf{u}}\langle\boldsymbol{\xi}\rangle$, $|\boldsymbol{\xi}| < \delta$, we write as $\mathbf{T}(\mathbf{x}_0)[\underline{\mathbf{u}}]\langle\boldsymbol{\xi}\rangle$, $|\boldsymbol{\xi}| < \delta$, is generated by the Fréchet derivative of the free energy. Thus, in the linearized theory presented here, $\mathbf{T}(\mathbf{x}_0)[\underline{\mathbf{u}}]\langle\boldsymbol{\xi}\rangle$ is defined by

$$\frac{d}{d\lambda} W(\mathbf{x}_0)[\underline{\mathbf{u}} + \lambda \underline{\mathbf{v}}]_{|\lambda=0} := \int_{\mathcal{H}_\delta(\mathbf{x}_0)} \mathbf{T}(\mathbf{x}_0)[\underline{\mathbf{u}}]\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{v}}\langle\boldsymbol{\xi}\rangle\, dv_{\boldsymbol{\xi}}\,, \quad \forall\, \underline{\mathbf{v}}\,. \tag{0.2}$$

Using the strain energy given by (0.1), we write $\mathbf{T}(\mathbf{x}_0)[\underline{\mathbf{u}}]\langle\boldsymbol{\xi}\rangle = \int_{\mathcal{H}_\delta(\mathbf{x}_0)} \mathbb{K}(\mathbf{x}_0)\langle\boldsymbol{\xi}, \boldsymbol{\eta}\rangle \underline{\mathbf{u}}(\boldsymbol{\eta})\, dv_{\boldsymbol{\eta}}$, where

$$\begin{aligned} \mathbb{K}(\mathbf{x}_0)\langle\boldsymbol{\xi}, \boldsymbol{\eta}\rangle \ :=\ & \frac{\omega(|\boldsymbol{\xi}|, |\boldsymbol{\eta}|)}{|\boldsymbol{\xi}||\boldsymbol{\eta}|} \Big[2\alpha_2 \big(\mathbf{e}(\boldsymbol{\xi}) \otimes \mathbf{e}(\boldsymbol{\eta})\big) + \frac{1}{2}\alpha_3 \big(\mathbf{e}(\boldsymbol{\xi}, \boldsymbol{\eta}) \otimes \mathbf{e}(\boldsymbol{\eta}, \boldsymbol{\xi})\big) \\ & + \frac{1}{2}\alpha_4 \big(\mathbf{e}(\boldsymbol{\xi}) \otimes \mathbf{e}(\boldsymbol{\eta}, \boldsymbol{\xi}) + \mathbf{e}(\boldsymbol{\xi}, \boldsymbol{\eta}) \otimes \mathbf{e}(\boldsymbol{\eta})\big) \Big] \\ & + \Delta(\boldsymbol{\eta} - \boldsymbol{\xi}) \int_{\mathcal{H}_\delta(\mathbf{x}_0)} \frac{\omega(|\boldsymbol{\zeta}|, |\boldsymbol{\eta}|)}{|\boldsymbol{\eta}|^2} \Big[\alpha_1 \mathbf{e}(\boldsymbol{\eta}) \otimes \mathbf{e}(\boldsymbol{\eta}) + \frac{1}{2}\alpha_3 \big(\mathbf{e}(\boldsymbol{\eta}, \boldsymbol{\zeta}) \otimes \mathbf{e}(\boldsymbol{\eta}, \boldsymbol{\zeta})\big) \\ & + \frac{1}{2}\alpha_4 \big(\mathbf{e}(\boldsymbol{\eta}) \otimes \mathbf{e}(\boldsymbol{\eta}, \boldsymbol{\zeta}) + \mathbf{e}(\boldsymbol{\eta}, \boldsymbol{\zeta}) \otimes \mathbf{e}(\boldsymbol{\eta})\big) \Big]\, dv_{\boldsymbol{\zeta}} \end{aligned} \tag{0.3}$$

is the modulus state for $|\boldsymbol{\xi}|, |\boldsymbol{\eta}| < \delta$, and $\Delta(\cdot)$ denotes the Dirac delta function.

In the limit of small horizon, Silling [1] shows that $C_{ijkl} = \int_{\mathcal{H}_\delta(\mathbf{x}_0)} \int_{\mathcal{H}_\delta(\mathbf{x}_0)} \mathbb{K}_{ik}(\mathbf{x}_0; \boldsymbol{\xi}, \boldsymbol{\eta})\, \xi_j\, \eta_l\, dV_\xi\, dV_\eta$, where C_{ijkl} are the elastic coefficients of a classical linear elastic material. Using (0.3) in these expressions, we find that the elastic coefficients satisfy the following relations: $C_{2313} = C_{1312} = C_{2312} = 0\,,\ C_{ii23} = C_{ii13} = C_{ii12} = 0, \quad i = 1, 2, 3\,,$ no sum on i . Furthermore, $\alpha_1 \cong 960.3\, (C_{1111} - C_{1122}) - 1920.6\, C_{1212}$, $\alpha_2 \cong -480.2\, (C_{1111} - C_{1122}) + 960.3\, C_{1212}$, $\alpha_3 \cong -1920.6\, (C_{1111} - C_{1122}) + 3841.2\, C_{1212}$, and α_4 is left as a free coefficient.

CONCLUSIONS

We have proposed a strain energy function for a linearly elastic peridynamic material that depends quadratically on linear measures of strain, which are analogous to measures of strain defined in the classical linear theory. The strain energy function contains four elastic peridynamic coefficients; three of which are given in terms of the elastic constants of the classical linear theory. The fourth coefficient is undetermined and is the subject of current investigation into the meaning of isotropy for peridynamic materials.

References

[1] Silling, S.A.: Linearized theory of peridynamic states, *J. Elasticity* **99**, 85-111, 2010.

[2] Silling, S.A., Epton, M., Weckner, O., Xu, J. & Askari, E., Peridynamic states and constitutive modeling, *J. Elasticity* **88**, 151-184, 2007.