EFFECTS OF LARGE SPECIFIC TORSION ANGLES, CONSTRAINED ANTICLASTIC CURVATURE AND SHEAR WARPING IN WIDE BEAMS

J. P. Meijaard*[a)]
Olton Engineering Consultancy, Enschede, The Netherlands

Summary The stiffness behaviour of beams with a large aspect ratio of their cross-sections, in particular leaf-springs, is considered. Boundary effects due to clamping for torsional, bending and shear loading and non-linear effects for large specific torsion angles and large curvatures due to flexion are shown. The influences of these effects on the stiffness are approximated by introducing variables that account for the amount of warping and anticlastic curvature. This makes the problems amenable to analytic solution methods. For the case of shear-flexible beams, this constitutes an extension of Timoshenko's beam theory.

INTRODUCTION

The stiffness behaviour of beams with a large aspect ratio of their cross-sections, in particular leaf-springs, is considered. These beams are characterized by a large ratio of their width b to their thickness t. The ratio of the length l and the thickness is still large, but the ratio l/b need not be large and can be of the order of unity. Therefore boundary effects at the ends of the beams due to clamping can be considerable. The clamping constrains the torsional warping, the anticlastic curvature and the warping due to shear, which influence the stiffness over boundary zones whose lengths are of the order of the width of a beam, which is not negligible in respect to its length. Furthermore, the effective flexural rigidity of the beam increases for large but realistic curvatures by a factor $1/(1-\nu^2)$, where ν is Poisson's ratio, due to a transition from a state of approximately plane stress to a state of approximately plane strain.

These effects can be included in linear or non-linear finite element calculations [1, 2]. For a better understanding of numerical results and for more efficient calculations in dynamic problems and during design and optimization, beam theories that include part of these effects in an approximate way are needed. These refinements are the subject of this paper.

TORSION

Constrained warping

The problem of constrained warping and non-uniform torsion can be considered solved [3, 4]. The used technique, where an additional variable is introduced to account for the amount of warping, can be used for other kinds of loading.

Large specific torsion angles and initial stresses

For large specific torsions, where the specific torsion angle is comparable with t/b^2, non-linear hardening of the torsional stiffness becomes important. This hardening results in the stable post-buckling behaviour of problems of torsional buckling. Related to this is the influence of initial stresses. Tensile stresses in the outer edges and compressive stresses near the centre increase the torsional stiffness, and, *vice versa*, compressive stresses near the edges and tensile stresses near the centre decrease the torsional stiffness.

ANTICLASTIC CURVATURE

Anticlastic curvature relates to the change in the shape of the cross-section of a beam in bending due to the transverse contraction. For a leaf-spring, a curvature κ of the mid-plane in the longitudinal direction is accompanied by a transverse curvature $\nu\kappa$ of the opposite sign.

Effects of boundary conditions

At clamped ends, the anticlastic curvature is prevented, and locally the flexural rigidity is increased by a factor $1/(1-\nu^2)$. This boundary effect damps out over a length of the order of the width of the beam. The effect is approximated by introducing the transverse curvature as an additional variable, which depends on the longitudinal distance from the boundary. Two coupled differential equations for the deflection and the transverse curvature are obtained. Some comparisons of their solutions with numerically obtained results [1] are made.

Effects of large curvatures

The bending of a strip into a circular cylinder has been considered by Lamb [5] and more precisely by Ashwell [6]. The result is that the anticlastic curvature is diminished for large longitudinal curvatures. The effect of clamping and cases of non-uniform curvature can be treated by the same approximation as was used for the boundary effects. This shows some significant differences from the results obtained by the use of the flexural rigidity for a uniform curvature.

[a)] Corresponding author. E-mail: J.P.Meijaard@olton.nl.

SHEAR

For loading in the plane of a leaf-spring, the contribution of shear to the flexibility becomes important if the ratio l/b is small. The effect of shear is usually included by means of Timoshenko's shear-flexible beam theory. This theory makes use of a shear correction factor [7], whose best value remains a subject of research. At a boundary where the beam is clamped, a shear correction value equal to one is locally valid, whereas away from the boundary, this factor is smaller. Here we introduce an additional variable that measures the amount of warping due to shear. In this way, the boundary perturbation is accounted for in an approximate way. Also the influence of a non-constant shear force, as can occur if a distributed loading is applied and in dynamic problems, can be taken into account. The results show that a single shear correction factor that is optimal for all load cases cannot be found.

CONCLUSIONS

Some effects that are important for beams with a large aspect ratio of their cross-sections, such as leaf-springs, can be analysed by analytic means. The distortion of the cross-section due to anticlastic curvature caused by bending has not often been included in analyses, but can be significant. An extension of Timoshenko's shear-flexible beam theory shows the influence of boundary conditions and non-uniform shear.

References

[1] Meijaard J.P.: Refinements of Classical Beam Theory for Beams With a Large Aspect Ratio of Their Cross-Sections. In Stépán G., Kovács L.L., Tóth A. (eds.): IUTAM Symposium on Dynamics Modeling and Interaction Control in Virtual and Real Environments, Springer, 2011, pp. 285–292.

[2] Brouwer D.M., Meijaard J.P., Jonker J.B.: Large Deflection Stiffness Analysis of Parallel Prismatic Leaf-Spring Flexures. Submitted, 2012.

[3] Timoschenko S.: On the Torsion of a Prism, One of the Cross-Sections of Which Remains Plane. *Proc. Lond. Math. Soc.* **20**:389–397, 1922.

[4] Vlasov V.Z: Thin-Walled Elastic Beams. Israel Program for Scientific Translations, Jerusalem, 1961.

[5] Lamb, H.: On the Flexure of a Flat Elastic Spring. *Phil. Mag. (5)* **31**:182–188, 1891.

[6] Ashwell D.G.: The Anticlastic Curvature of Rectangular Beams and Plates. *J. Roy. Aer. Soc.* **54**:708–715, 1950.

[7] Cowper G.R.: The Shear Coefficient in Timoshenko's Beam Theory. *ASME J. Appl. Mech.* **33**:335–340, 1966.