INTEGRAL IDENTITIES FOR A SEMI-INFINITE INTERFACIAL CRACK IN 2D AND 3D ELASTICITY

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Summary The paper is concerned with the problem of a semi-infinite crack at the interface between two dissimilar elastic half-spaces, loaded by a general asymmetrical system of forces distributed along the crack faces. On the basis of the weight function approach and the fundamental reciprocal identity (Betti formula), we formulate the elasticity problem in terms of singular integral equations relating the applied loading and the resulting crack opening. Such formulation is fundamental in the theory of elasticity and extensively used to solve several problems in linear elastic fracture mechanics (for instance various classic crack problems in homogeneous and heterogeneous media). This formulation is also crucial in important recent multiphysics applications, where the elastic problem is coupled with other concurrent physical phenomena.

INTRODUCTION

Three-dimensional problems of linear elasticity can be solved nowadays by a variety of techniques. Purely numerical methods, such as finite element and boundary element techniques, are very popular and extensively used, thanks to the computational power of modern computers. However, analytical methods, such as integral transforms, variational inequalities, and singular integral equations, are still of great interest, for instance in deriving fundamental solutions for some basic geometries (to be employed in numerical methods), in the assertion of existence and uniqueness of the solution, and in bifurcation theory (Bigoni and Capuani, 2002).

In particular, the method of singular integral equations was first developed for two-dimensional problems and later extended to three-dimensional problems by the rapidly developing theory of multi-dimensional singular integral operators (Kupradze et al., 1979). For crack problems, the general approach to derive the singular integral formulation is based on the Green's function method (Weaver, 1977). This method consists in two steps. First, the stresses at a point $\boldsymbol{x}$ inside a body containing a discontinuity surface S are expressed in the form

$$\sigma_{ij}(\boldsymbol{x}) = \int_S K_{ijk}(\boldsymbol{x},\boldsymbol{\xi})\Delta u_k(\boldsymbol{\xi})d\boldsymbol{\xi}, \tag{1}$$

where $K_{ijk}(\boldsymbol{x},\boldsymbol{\xi}) = c_{ijst}(\partial/\partial x_t)T^+_{sk}(\boldsymbol{x},\boldsymbol{\xi})$ and $T^+_{sk}(\boldsymbol{x},\boldsymbol{\xi})$ is the traction associated with the Green's function $U_{sp}(\boldsymbol{x},\boldsymbol{\xi})$, according to the equation $T^+_{sk}(\boldsymbol{x},\boldsymbol{\xi}) = n^+_l c_{klpq}U_{sp,q}(\boldsymbol{x},\boldsymbol{\xi})$. Here c_{ijst} is the elasticity tensor, n^+_l is the outward normal on the $+$ side of the surface and the displacement discontinuity is denoted by $\Delta u_k = u^-_k - u^+_k$. The second step consists in taking the limit as the point $\boldsymbol{x}$ approaches a point $\boldsymbol{x}_0$ on the surface S, to obtain the tractions on the surface

$$p^+_i(\boldsymbol{x}_0) = \lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\int_S n^+_j(\boldsymbol{x})K_{ijk}(\boldsymbol{x},\boldsymbol{\xi})\Delta u_k(\boldsymbol{\xi})d\boldsymbol{\xi}. \tag{2}$$

The described procedure works when the tractions on the discontinuity surface S are symmetrical. Moreover, the limit in (2) is not streigthforward and due care should be taken to ensure the convergence of the integrals.

In this paper, we consider the three-dimensional problem of a semi-infinite crack at the interface between two dissimilar elastic half-spaces and apply a method which avoids the use of the Green's function and the tedious limiting procedure. We also do not use the assumption of symmetrical load.

Technique

Our results are based on integral transforms and two fundamental notions of linear elasticity: the Betti reciprocal theorem, on the one hand, and the weight function approach, on the other. The Betti identity has been extensively used in the perturbation analysis of two and three-dimensional cracks (Piccolroaz et al., 2007). The concept of weight function was introduced in linear fracture mechanics by Bueckner (1970).

Assumptions

We consider a semi-infinite crack at the interface between two dissimilar elastic half-spaces $\mathbb{R}^3_\pm = \{\boldsymbol{x} = (x_1,x_2,x_3) \in \mathbb{R}^3 : \pm x_2 > 0\}$, see Fig. 1. The two elastic half-spaces are assumed to be isotropic with shear modulus and Poisson's ratio denoted by $\mu_\pm$ and $\nu_\pm$, respectively. The crack lies on the half-plane $\mathbb{R}^2_- = \{\boldsymbol{x} = (x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 < 0, x_2 = 0\}$. The crack faces are loaded by a system of, not necessarily symmetrical, distributed forces $\boldsymbol{p}_\pm$. It is convenient to introduce the symmetrical and skew-symmetrical parts of the loading as follows

$$\langle \boldsymbol{p}\rangle = \frac{1}{2}(\boldsymbol{p}_+ + \boldsymbol{p}_-), \quad [\![\boldsymbol{p}]\!] = \boldsymbol{p}_+ - \boldsymbol{p}_-, \tag{3}$$

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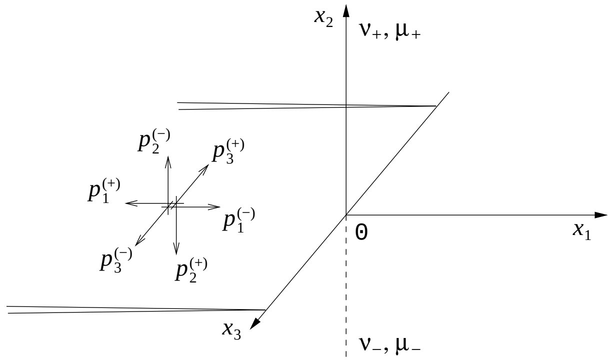

Figure 1. Semi-infinite interfacial crack loaded by asymmetrical forces on the crack faces.

where we used standard notations to denote the average, $\langle f \rangle$, and the jump, $[\![f]\!]$, of a function f across the plane containing the crack, $x_2 = 0$,

$$\langle f \rangle(x_1, x_3) = \frac{1}{2}[f(x_1, 0^+, x_3) + f(x_1, 0^-, x_3)], \quad [\![f]\!](x_1, x_3) = f(x_1, 0^+, x_3) - f(x_1, 0^-, x_3). \tag{4}$$

RESULTS

Antiplane shear

For antiplane shear deformation, the integral identities read

$$\langle p_3 \rangle + \frac{\eta}{2}[\![p_3]\!] = -\frac{1}{b+e}\mathcal{S}^{(s)}\frac{\partial [\![u_3]\!]^{(-)}}{\partial x_1} \quad \text{for} \quad x_1 < 0, \quad \langle \sigma_{23} \rangle^+ = -\frac{1}{b+e}\mathcal{S}^{(c)}\frac{\partial [\![u_3]\!]^{(-)}}{\partial x_1} \quad \text{for} \quad x_1 > 0, \tag{5}$$

where $\mathcal{S}^{(s)}$ is a singular operator, whereas $\mathcal{S}^{(c)}$ is a compact operator,

$$\mathcal{S}^{(s)}\varphi = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\varphi(\xi)}{x_1 - \xi}d\xi \quad \text{for} \quad x_1 < 0, \quad \mathcal{S}^{(c)}\varphi = \frac{1}{\pi}\int_{-\infty}^{0}\frac{\varphi(\xi)}{x_1 - \xi}d\xi \quad \text{for} \quad x_1 > 0. \tag{6}$$

Plane strain and general 3D case

For plane strain and general 3D case, the integral identities can be written in the matrix form

$$\langle \boldsymbol{p} \rangle + \boldsymbol{\mathcal{A}}^{(s)}[\![\boldsymbol{p}]\!] = \boldsymbol{\mathcal{B}}^{(s)}[\![\boldsymbol{u}]\!]^{(-)} \quad \text{for} \quad x_1 < 0, \tag{7}$$

$$\langle \boldsymbol{\sigma}_2 \rangle^{(+)} + \boldsymbol{\mathcal{A}}^{(c)}[\![\boldsymbol{p}]\!] = \boldsymbol{\mathcal{B}}^{(c)}[\![\boldsymbol{u}]\!]^{(-)} \quad \text{for} \quad x_1 > 0, \tag{8}$$

where $\boldsymbol{\mathcal{A}}^{(s)}$ and $\boldsymbol{\mathcal{B}}^{(s)}$ are singular matrix operators, whereas $\boldsymbol{\mathcal{A}}^{(c)}$ and $\boldsymbol{\mathcal{B}}^{(c)}$ are compact ones (for the definition of these operators see Piccolroaz and Mishuris, 2011).

CONCLUSIONS

The singular integral formulation for a three-dimensional semi-infinite interfacial crack has been derived in explicit form by means of fundamental properties of linear elasticity and integral transforms. This formulation for interfacial cracks seems to be unknown in the literature. The method of derivation avoids the use of Green's function and the tedious limiting process involved in the general procedure. The identities are important in applications especially in case of multiphysics problems where the elasticity equations are coupled with other concurrent phenomena, such as in hydraulic fracturing.

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