AN EQUILIBRIUM PROBLEMS OF A RECTANGLE IN THE COSSERAT ELASTICITY

Grigoriev Yuri M.[a)]
Department of Theoretical Physics, North-Eastern Federal University named after M.K. Ammosov, Yakuts, 677000, Russia

Summary The method of an exact analytical solution of an equilibrium problems of a rectangle in two dimensional micropolar elasticity theory (Cosserat continuum) is developed. Conceded problems can be decomposed into a number of boundary value problems of the certain symmetry which solution after using of theorems of continuation, cam be reduced to the problems with the mixed boundary conditions. Such mixed boundary conditions allow to obtain the problems of the Dirichlet or the Neumann for an auxiliary function that is proportional to a divergence of a displacement. For other unknown functions (components of rotation and displacement) the Riquier type problems, problems for the Poisson and Helmholtz equations with mixed boundary conditions must be solved. Examples of solutions of problems by mean of the given method are obtained. Numerical realizations and analysis of the obtained solutions are worked out. Obtained exact analytical solutions can be used for developing a method for measuring the micropolar elastic constants. The method can be generalized to a three-dimensional case and on the harmonic oscillation problems.

INTRODUCTION

Experimental results of solid mechanics show that the mechanical behavior of many materials used in practice can not be theoretically explained by means of classical continuum mechanics. There is a need to use models with internal degrees of freedom. The same situation is in geomechanics. Of models with internal degrees of freedom in the theory of elasticity is the most developed micropolar (Kosserat) elasticity theory [1]. The basis of this model is the idea of a Cosserat continuum, where elementary particle of continuum is endowed with six degrees of freedom. Generalization of this theory for anisotropy, heterogeneity, related fields, the theory of plates and shells, fracture are developed [2]. A search for new analytical solutions to the Cosserat theory, a study the wave properties of micropolar media are doing. There is a progress in measuring the elastic constants of the micropolar media and in understanding of the mechanism of the couple stresses. The forces and moments acting on an arbitrary plane section environment are obtained by averaging over this section of the microstresses and mikromoments. If these non-uniform microfield, then there are nonzero average coupled stresses. In formulating the boundary conditions that adequately take into account the roughness, the friction and wear, to be considered include the coupled stresses. Many of the difficulties of the development and application of micropolar theory of elasticity due to the lack of generally accepted method of experimental measurement of all six of the elastic constants. This is due to the lack of a sufficient number of exact analytical solutions. In this paper we develop a new method for obtaining exact analytical solutions of problems of equilibrium and the harmonic vibrations of a rectangle in the micropolar elasticity theory. Based on these solutions and those of similar three-dimensional results a method of experimental measurement of all six micropolar elastic constants can be developed.

BASIC EQUATIONS

The equilibrium equations of the micropolar theory of elasticity are the next form:

$$\begin{cases}(\mu+\alpha)\Delta\mathbf{u}+(\lambda+\mu-\alpha)\nabla(\nabla\cdot\mathbf{u})+2\alpha\nabla\times\boldsymbol{\omega}=0,\\(\nu+\beta)\Delta\boldsymbol{\omega}+(\varepsilon+\nu-\beta)\nabla(\nabla\cdot\boldsymbol{\omega})+2\alpha\nabla\times\mathbf{u}-4\alpha\boldsymbol{\omega}=0.\end{cases}$$

We consider plane deformations when displacement vector $\mathbf{u}$ and the rotation vectors $\boldsymbol{\omega}$ of medium points have the forms:

$$\mathbf{u}=u_x(x,y)\mathbf{i}+u_y(x,y)\mathbf{j},\qquad \boldsymbol{\omega}=\omega(x,y)\mathbf{k}.$$

Also one must to add the Hooke's low relations with linear relations between stress σ_{ij}, coupled stress μ_{ij} components and strain, microrotations components [1].

Boundary conditions

For every boundary conditions by splitting of a solution into symmetric and antisymmetric parts one can obtain a number of boundary problems of the certain symmetry. Using of theorems of continuation of solutions through a linear part of boundary into the mirror reflected area [3] such problems can be reduced to the problems with mixed boundary conditions. For example consider the next:

a) Grigoriev Yuri M. Email: grigyum@yandex.ru

$$\begin{cases} \sigma_{xx}\big|_{x=a,0} = s_x^{\pm}(y), \quad u_y\big|_{x=a,0} = d_y^{\pm}(y), \; \mu_{xz}\big|_{x=a,0} = m_x^{\pm}(y), \\ \sigma_{yy}\big|_{y=b,0} = s_y^{\pm}(x), \quad u_x\big|_{y=b,0} = d_x^{\pm}(x), \; \mu_{yz}\big|_{y=b,0} = m_y^{\pm}(x), \end{cases} \tag{1}$$

for the rectangle $\Omega = \{0 < x < a, 0 < y < b\}$. We assume that the principal moment of the applied coupled stress is zero, all boundary functions are compatible on each corners of the rectangle and displacements and rotations are enough differentiable in the closed domain. Existing and uniqueness theorems for all boundary value problems in the micropolar elasticity are well known.

THE METHOD OF SOLUTION

From Hooke's low relations and given boundary conditions (1) one can to obtain the boundary conditions for the introduced auxiliary function $f = (\lambda + 2\mu)\nabla \cdot \mathbf{u}$ [3]. From equilibrium equations follows that this function is harmonic of this function. Thus, we have the Dirichlet problem:

$$\Delta f = 0, \mathbf{r} \in \Pi, \qquad f\big|_{x=a,0} = s_x^{\pm}(y) + 2\mu d_y^{\pm\prime}(y), \qquad f\big|_{y=b,0} = s_y^{\pm}(x) + 2\mu d_x^{\pm\prime}(x).$$

Then for the rotation ω we have the Riquier-type problem with the equation $\Delta(\Delta - k^2)\omega = 0$, that can be derived from the equilibrium equations. Boundary conditions also can be derived from the equilibrium equations, Hooke's low and the original boundary conditions (1). After solving these problems for displacement components we have the mixed boundary value problems for the Poisson equation.

EXAMPLE

For examples let us consider special case in the boundary conditions (1) when all boundary functions are zero and only one is nonzero: $s_x^{+} = -\sigma \sin(\pi y / b)$. In this case by using Fourier expansion we have obtained the next solution:

$$\omega = \frac{\sigma}{2\mu}\left(\frac{\operatorname{ch}\frac{\pi x}{b}}{\operatorname{sh}\frac{\pi a}{b}} - \frac{\pi}{b\theta}\frac{\operatorname{ch}\theta x}{\operatorname{sh}\theta a} \right)\cos\frac{\pi y}{b}, \qquad \theta^2 = k^2 + \frac{\pi^2}{b^2} = \frac{4\alpha\mu}{B(\mu+\alpha)} + \frac{\pi^2}{b^2} \tag{2}$$

$$u_x = \left[\frac{\lambda+\mu}{\lambda+2\mu}\frac{\pi}{b}\frac{x\operatorname{sh}\frac{\pi x}{b}}{\operatorname{sh}\frac{\pi a}{b}} + \left(\frac{B}{2\mu} - \frac{\lambda+\mu}{\lambda+2\mu}\frac{b}{\pi} a \operatorname{cth}\frac{\pi a}{b} - \frac{\lambda+3\mu}{\lambda+2\mu}\frac{b^2}{\pi^2} \right)\frac{\operatorname{ch}\frac{\pi x}{b}}{\operatorname{sh}\frac{\pi a}{b}} - \frac{B}{2\mu}\frac{\pi}{b\theta}\frac{\operatorname{ch}\theta x}{\operatorname{sh}\theta a} \right] \cdot \frac{\sigma}{2\mu}\frac{\pi}{b}\sin\frac{\pi y}{b},$$

$$u_y = \left[\frac{\lambda+\mu}{\lambda+2\mu}\frac{\pi}{b}\frac{x\operatorname{ch}\frac{\pi x}{b}}{\operatorname{sh}\frac{\pi a}{b}} + \left(\frac{B}{2\mu} - \frac{\lambda+\mu}{\lambda+2\mu}\frac{b}{\pi} a \operatorname{cth}\frac{\pi a}{b} \right)\frac{\operatorname{sh}\frac{\pi x}{b}}{\operatorname{sh}\frac{\pi a}{b}} - \frac{B}{2\mu}\frac{\operatorname{sh}\theta x}{\operatorname{sh}\theta a} \right]\frac{\sigma}{2\mu}\frac{\pi}{b}\cos\frac{\pi y}{b}.$$

Here λ, μ, α, B are the material micropolar constants. In the case when $\alpha \to 0$ from (2) we have solution classical solution. Numerical analysis was made by means of taken formula (2). Other examples have obtained by the presented method.

CONCLUSION

The developed method allows us to solve some problem of equilibrium of a rectangle in the micropolar elasticity in exact analytical form. Obtained exact analytical solutions can be used for developing a method for measuring the micropolar elastic constants. The method can be generalized to three-dimensional case and to harmonic oscillation problems.

References

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