A POINT-FORCE ELASTIC PROBLEM FOR EXPONENTIAL FUNCTIONALLY GRADED COATINGS

Roberta Sburlati[a)]
Department of Civil, Environmental and Architectural Engineering, DICAT, 16145, Genoa, Italy
Research Center for Materials Science and Technology, MaST University of Genoa, 16145, Genoa, Italy

Summary The aim of this contribution is to study the three-dimensional elastic deformation of a functionally graded coating deposited on a homogenous substrate subjected to a point load. The solution is obtained in the framework of the three-dimensional elasticity and it can be useful to describe the inhomogeneity effects on the elastic response and investigate the presence of damage at the interface between the coating and the substrate.

INTRODUCTION

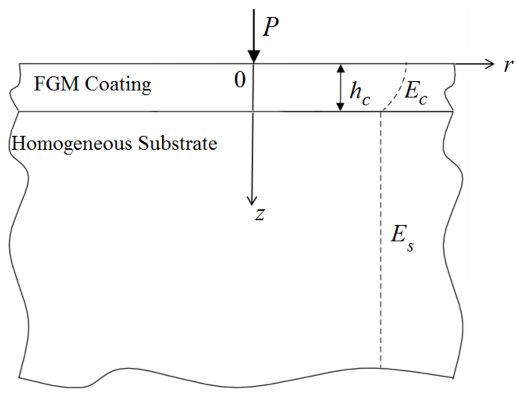
Functionally Graded Materials (FGMs) are composites with continuously varying volume fractions of their constituent homogeneous phases. If the characteristic length scale of such variation is much larger than the size of the phases, the model of the inhomogeneous solids is able to describe the macroscopic behaviour of these materials by means of homogenization techniques. By considering materials in which this assumption is possible, we obtain an elastic solution in the framework of the three-dimensional elasticity, for the point load problem of a FGM coating deposited on a homogenous substrate. By using the Plevako's representation form able to reduce our problem to the search of a potential function satisfying a linear fourth-order partial differential equation, we explicitly get the solution by writing the potential function with a Bessel expansion with respect to the radial coordinate [1,2]. The method of solution is successfully used to obtain benchmark solutions to validate numerical solutions or results obtained by structural theories [3,4,5].

STATEMENT OF THE PROBLEM AND METHOD OF SOLUTION

We consider a functionally graded material coating of thickness h_c deposited on a homogeneous substrate and subjected to a point load on the upper face. By introducing a cylindrical coordinate system, we assume that in the coating, the Young's modulus varies exponentially in the z-direction, while the Poisson ratio ν_c is uniform. The substrate is homogeneous and isotropic with elastic properties E_s and $\nu_s = \nu_c$:

$$E_c = E_0 e^{2kz} \quad 0 \le z \le h_c \ , \qquad E_s = E_0 e^{2kh_c} \quad z > h_c \qquad \text{and} \qquad \nu_c = \nu_s = \nu$$

The problem is studied for two different interface conditions between the coating and the substrate: frictionless contact and perfectly bonded (Figure 1).



Boundary and interface conditions

$$\sigma_z^{(c)}(r,0) = -\frac{P\delta(r)}{\pi r}\ , \qquad \tau_{rz}^{(c)}(r,0) = 0;$$

$$\lim_{z\to\infty} \sigma_z^{(s)}(r,z) = 0, \qquad \lim_{z\to\infty} \tau_{rz}^{(s)}(r,z) = 0;$$

-perfectly bonded:

$$u_r^{(c)}(r,h_c) = u_r^{(s)}(r,h_c), \qquad u_z^{(c)}(r,h_c) = u_z^{(s)}(r,h_c),$$

$$\sigma_z^{(c)}(r,h_c) = \sigma_z^{(s)}(r,h_c), \qquad \tau_{rz}^{(c)}(r,h_c) = \tau_{rz}^{(s)}(r,h_c);$$

-frictionless contact:

$$u_z^{(c)}(r,h_c) = u_z^{(s)}(r,h_c),$$

$$\sigma_z^{(c)}(r,h_c) = \sigma_z^{(s)}(r,h_c), \qquad \tau_{rz}^{(c)}(r,h_c) = \tau_{rz}^{(s)}(r,h_c) = 0.$$

Figure 1: Geometry, boundary and interface conditions of the point-force elastic problem

In order to find the elastic solution for this problem, we follow the Plevako's approach for the axisymmetric case; the displacement field is expressed in terms of potential functions $L^{(i)}(r, z)$, respectively for the FGM coating (i=c) and for the homogeneous substrate (i=s) [1].

a) Corresponding author. Email: sburlati@dicat.unige.it

We obtain the following different partial differential equations

$$\nabla_r^4 L^{(c)} + 2\left(\frac{\partial^2}{\partial z^2} - 2k\frac{\partial}{\partial z} - 2k^2\omega^2\right)\nabla_r^2 L^{(c)} + \left(\frac{\partial^2}{\partial z^2} - 4k\frac{\partial}{\partial z} + 4k^2\right)\frac{\partial^2}{\partial z^2} L^{(c)} = 0, \qquad \nabla_r^4 L^{(s)} + 2\frac{\partial^2}{\partial z^2}\nabla_r^2 L^{(s)} + \frac{\partial^4}{\partial z^4} L^{(s)} = 0,$$

where ∇_r^2 denotes the radial the radial Laplace operator and $\omega^2 = \nu/(1-\nu)$. Now, we assume that there exists a cylindrical region of radius $b >> h_c$ where the elastic deformation is bonded; consequently, we suppose that for every $r > b$, the transversal displacement field is null. This assumption permits to write the potential functions, describing the elastic response in the coating and in the substrate, in terms of the Fourier-Bessel expansion

$$L^{(i)}(r,z) = \sum_{j=1}^{n} L_j^{(i)}(z)\, J_0(\phi_j r) \quad \text{where} \quad \phi_j = \frac{z_j^{(0)}}{b}, \qquad L_j^{(i)}(z) = \frac{2}{b^2 J_1^2(\phi_j b)} \int_0^b L^{(i)}(\rho, z)\, J_0(\phi_j \rho)\,\rho\, d\rho,$$

and $z_j^{(0)}$ are the positive roots of the zero-order Bessel J-function [6]. The elastic solution is so obtained in terms of eight series of the coefficients which can be fully determined by using the boundary and the interface conditions.

APPLICATION AND CONCLUSIONS

In order to examine the different effects of the graded properties and different interface conditions, we conduct a comparative study of a functionally graded coating (*FGMc*) versus a homogeneous coating (*HMc*). We choose a hardened surface (k<0) with a quite severe Young's modulus gradation E_0 =100E_s and a uniform Poisson ratio $\nu = 0.3$; further, we consider a load P=1MPa and a ratio b/h_c=10. In the homogeneous coating and substrate case we assume: E_c=100E_s. In Figure 2, we report the radial displacement near the point-force $\left(r = 1/15\, h_c\right)$ throughout the thickness of the coating; we observe the increasing of the radial displacement in the graded coating thickness in comparison with the homogeneous coating case and, in the interface, we note an increasing of the gap of the displacement corresponding to the frictionless contact interface condition.

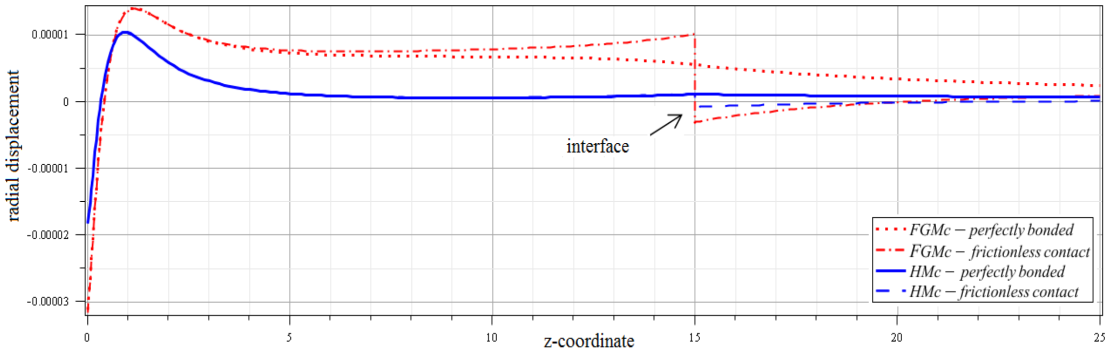


Figure 2: Comparison of the radial displacements in the thickness of the coating and the substrate

Investigations on the stresses, with respect to the corresponding homogeneous coating, reveal an increasing of the hoop and radial stress on the free surface and a reduction on the interface zone due to the graded properties. The different interface conditions that we analyzed describe two ideal cases able to bound the real cases where the contact between the coating and the substrate are neither frictionless nor perfectly bonded; in this way, these limit cases permit to study the effects of localized interface damage. Finally, the exact analytical solution so obtained permits to conduct further parametric analyses to provide useful suggestions for the design of functionally graded coatings.

Acknowledgements

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