ON VARIATIONAL DIMENSION-REDUCTION METHODS TO DERIVE SHELL THEORIES

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Summary The classic models for elastic structures are based on certain a priori assumptions on the deformation or stress fields, diverse in nature but all motivated by the smallness of certain dimensions with respect to others. In the last years, a considerable amount of work has been done in order to motivate, interpret and classify such assumptions within the framework of three-dimensional elasticity, linear and not; and, several analytical methods have been introduced to carry out rigorous dimension reductions. In this paper, I concentrate on a modeling idea – that of imposing fiber-wise constraints to model the behavior of plates and shells – that efficiently replaces a priori assumptions on the admissible deformation of Reissner-Mindlin type, especially when in connection with a nonstandard approach to variational convergence.

INTRODUCTION

All classic models for thin elastic structures admit a variational formulation, consisting in the minimization of a peculiar energy functional F^a on a peculiar function space H^a. Problem P^a:

$$\text{find} \quad \arg\min_{v\in H^a} F^a(v),$$

is meant to furnish a reasonable approximation of the solution of a real problem P^r, at a reasonable computational cost. To quantify what is reasonable is a question that is best examined case by case, given that a conceptual validation of the model at hand has been achieved. A fashionable method to validate the classic models is to establish by Γ-convergence [1, 2] the position of their 'lower-dimensional' energy functionals with respect to an energy functional F^r whose form is chosen according to some three-dimensional parent theory of elasticity. With Γ-convergence, *problem convergence implies solution convergence*: given a sequence $\{P_\varepsilon\}$ of problems and the corresponding sequence $\{u_\varepsilon\}$ of solutions, the variational convergence of $\{P_\varepsilon\}$ to a limit problem P_0 implies the convergence of $\{u_\varepsilon\}$ to a solution u_0 of P_0:

$$P_\varepsilon \to P_0 \;\Rightarrow\; u_\varepsilon \to u_0.$$

A relevant issue is whether and how the limit problem is related to the real problem of interest: given P^r, a variationally convergent sequence $\{P_\varepsilon\}$ has to be found, such that its limit problem P_0 is in tight kinship with P^a, and hence with P^r. When the parent theory is linear elasticity, a method to produce an exahustive collection of $(\{F_\varepsilon\}, F_0)$ pairs to be associated to both two- and one-dimensional structure models has been proposed by Miara and Podio-Guidugli in [3]. Prompted by the results in [3], Paroni, Podio-Guidugli, and Tomassetti [4] have produced a sequence $\{F_\varepsilon\}$ of energy functionals of three-dimensional linear elasticity that $\Gamma-$converges to a limit functional F_0, with F_0 reducing to Reissner-Mindlin two-dimensional functional F^a_{RM} via thickness integration, when evaluated over a Reissner-Mindlin displacement field. More recently, Percivale and Podio-Guidugli [5] have shown how to incorporate the kinematical features relevant for general plate models of Reissner-Mindlin type into the definition of the function space over which the energy functionals are defined. In this paper, I concentrate on the modeling idea exploited in [5] – namely, that of imposing fiber-wise constraints to model the behavior of plates and shells – in connection with a not completely standard use of variational convergence that is the subject on ongoing research with R. Paroni [6]. Given the allotted space, in the following I limit myself to sketch three issues that are at the core of my developments: (i) the notions of point- and fiber-wise constraints; (ii) how the latter are formulated for shell structures; (iii) classical and nonclassical problem sequences.

POINT-WISE AND FIBER-WISE CONSTRAINTS

The R-M Ansatz is:

$$\mathbf{u}_{RM}(x, x_3) = \mathbf{v}(x) + w(x)\mathbf{e}_3 + \zeta\,\boldsymbol{\varphi}(x) = (v_\alpha(x) + x_3\varphi_\alpha(x))\mathbf{e}_\alpha + w(x)\mathbf{e}_3, \tag{1}$$

an a priori representation parameterized by five fields over a two-dimensional flat region interpreted as a plate's cross section. This a priori representation - which is quickly shown equivalent to requiring that transverse fibers are inextensible, remain straight, and deform homogeneously - may be seen as of the combination of two *internal constraints*:

$$u_{3,3} = 0 \quad \text{and} \quad \mathbf{u}_{,33} = 0, \tag{2}$$

of the first and second order, respectively, both imposed pointwise. Now, in variational elasticity, there are two not mutually exclusive ways to take a constraint into account: either it is made part of the definition of the function space over which the functional to minimize is defined or that functional is penalized by addition of a term that vanishes only if the constraint in question is enforced. In [4], to take the second of conditions (2) into account, the latter way was followed.

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In this paper, just as was done first in [5], we follow the former way in an unconventional manner, in that we replace that second-order point constraint with a suitable *first-order constraint imposed fiberwise*:

$$^s(\mathbf{R}\cdot\mathbf{E})=\mathbf{0},\quad \int R^{\alpha 3}(x,\zeta)E_{\alpha 3}(x,\zeta)\,\alpha(x,\zeta)\,d\zeta=0; \tag{3}$$

here $\mathbf{R}$ and $\mathbf{E}$ denote, respectively, the reactive stress field that maintains the constraint and the strain field, while $\alpha(x,\zeta)$ adjourns the area measure along the fiber $\mathcal{F}(x)$ through x (see the next section). It would seem that fiber constraints are the 'natural' ones to use in the case of plate and shell theories, just as cross-section constraints are 'natural' in the case of straight and curved rod theories.

A FEW BITS OF SHELL GEOMETRY

The geometry of a three-dimensional *shell-like region* is dictated by the geometry of the base surface that serves to model it: to an architect or a structure engineer, the very first perception of a shell structure consists in a visualization of such a surface from the combined points of view of esthetics and statics. For simplicity, the typical base surface $\mathcal{S}$ admits a smooth global parametrization without exceptions, and the associated shell-like region is a *tubular neighborhood* $\mathcal{G}(h)$ of $\mathcal{S}$ of constant thickness $2h$; under the assumptions, $\mathcal{G}(h)$ too admits a global parametrization [7]. Along a fiber $\mathcal{F}(x):=\{p=x+\zeta\,\mathbf{n}(x)\in\mathcal{G}(h)\,|\,(p-x)\times\mathbf{n}(x)=\mathbf{0}\}$ (here $\mathbf{n}(x)$ denotes the value at $x\in\mathcal{S}$ of the *normal field*). The covariant basis is $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{n})$ at x and $(\mathbf{g}_1,\mathbf{g}_2,\mathbf{n})$ at p, with $\mathbf{g}_\alpha=\mathbf{e}_\alpha+\zeta\,\mathbf{n}_{,\alpha}$; for $(\mathbf{e}^1,\mathbf{e}^2,\mathbf{n})$, with $\mathbf{e}^\alpha:=\partial_x\zeta^\alpha$, the contravariant basis at x. The *shifter* $\mathbf{A}:=\mathbf{g}_i\otimes\mathbf{e}^i=\mathbf{g}_\alpha\otimes\mathbf{e}^\alpha+\mathbf{n}\otimes\mathbf{n}=:{}^s\mathbf{A}+\mathbf{n}\otimes\mathbf{n}$ maps linearly the covariant basis at x into the covariant basis at p. In particular, the *superficial shifter* ${}^s\mathbf{A}$ is a linear transformation of the tangent space to $\mathcal{S}$ at x into the span of $(\mathbf{g}_1,\mathbf{g}_2)$ at p; interestingly, ${}^s\mathbf{A}={}^s\mathbf{1}-\zeta\,\mathbf{K}$, $\quad{}^s\mathbf{1}:=\mathbf{e}_\alpha\otimes\mathbf{e}^\alpha$, $\quad\mathbf{K}:=-{}^s\nabla\mathbf{n}=-\mathbf{n}_{,\alpha}\otimes\mathbf{e}^\alpha$, where ${}^s\mathbf{1}$ is the *identity tensor* and $\mathbf{K}$ is Weingarten's *curvature tensor* of the base surface $\mathcal{S}$. The jacobian $\alpha:=\det\mathbf{A}$ can be computed in terms of the *average* and *gaussian curvatures* κ_a and κ_g of $\mathcal{S}$: $\alpha(x,\zeta)=1-\zeta\,\kappa_m(x)+\zeta^2\kappa_g(x)$, $\kappa_m:=\operatorname{tr}\mathbf{K}$, $\kappa_g:=\det\mathbf{K}$. The following integration formulae hold for a field $\Psi(p)\equiv\Psi(x,\zeta)$ defined over $\mathcal{G}(h)$:

$$\int_{\mathcal{P}_{\mathcal{G}(h)}}\Psi(p)\,dv|_p=\int_{\mathcal{P}_\mathcal{S}}{}^s\Psi(x)\,da|_x,\quad {}^s\Psi(x):=\int_{\mathcal{F}(x)}\Psi(x,\zeta)\,\alpha(x,\zeta)\,d\zeta, \tag{4}$$

where, given the part $\mathcal{P}_\mathcal{S}$ of $\mathcal{S}$, $\mathcal{P}_{\mathcal{G}(h)}$ is the corresponding part of $\mathcal{G}(h)$ and ${}^s\Psi$ is the *fiber average* of the field Ψ. Note that (3) is nothing but an application of this formula, with $\Psi(x,\zeta)=R^{\alpha 3}(x,\zeta)E_{\alpha 3}(x,\zeta)$. Needless to say, when the base surface is flat, the above geometric structure trivializes: the tensorial fields ${}^s\mathbf{1}(x)$ and $\mathbf{K}(x)$, that characterize $\mathcal{S}$ locally up to the second order, become ${}^s\mathbf{1}(x)\equiv{}^s\mathbf{1}$ and $\mathbf{K}(x)\equiv\mathbf{0}$; consequently, $\alpha(x,\zeta)\equiv 1$. This should be kept in mind when adapting to truly shell-like regions procedures and analytical developments taken from [5].

CLASSICAL AND NONCLASSICAL PROBLEM SEQUENCES

Problem sequences $\{P_\varepsilon\}$ are classically constructed in two steps: (i) an omothetical sequence of domains $\Omega_\varepsilon=\omega\times\varepsilon(-1,+1)$, $\varepsilon\in(0,1]$, is chosen, with ω a fixed two-dimensional region (here, $\omega\equiv\mathcal{S}$) and with $\Omega^r=\Omega_{\varepsilon^r}$ being the three-dimensional domain where the real problem P^r is formulated; (ii) for each domain Ω_ε, a functional F_ε is defined, under the form of an integral over Ω_ε of an integrand that may or may not depend on ε, to be minimized over a space of H^1 functions that vanish over the Dirichlet part of the boundary of Ω_ε; the sequence $\{F_\varepsilon\}$ has to be such that $\{P_\varepsilon\}$ variationally converges and that $P_{\varepsilon^r}=P^r$. Because of variational convergence, $\{u_\varepsilon\}$, the sequence of solutions to problems P_ε, is for ε small close to u_0, the solution of problem P_0. But, provided that ε^r too is chosen small, the solution u^r of P^r is equal to u_{ε^r} and hence it is also close to u_0. This reasoning 'proves' that the solution of problem P_0 is close to the solution of problem P^r. As argued in [6], with the exception of $\varepsilon=\varepsilon^r$, all problems P_ε may well be 'artificial': what counts is that P_0, at times another 'artificial' problem, is judged reasonably 'close' to P^r, possibly because it is 'close' to an accepted model problem P^a. Now, there may be more than one sequence satisfying the properties above and hence more than a limit problem, both 'closed' to P^r, each of which captures some but in general not all of the essential features of the real solution u^r, perhaps with different degrees of accuracy. Physical intuition should be used in the construction of a problem sequence, because none could be chosen a priori on purely mathematical grounds.

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