BAND-GAP OF EIGENFREQUENCY DUE TO NONLOCALITY IN ELASTIC STRING

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Summary The Lagrangian formulations of continuum with nonlocality are used to investigate the vibration of an elastic string. We find that there exists the band-gap of frequency in the vibration of the elastic string without periodic micro-structures if the long-range interaction within the string is concerned. This result shows that the long-range interaction is probably a new mechanism to form the band-gap of frequency in amorphous materials.

Basic equations and results

Phononic crystals are artificially fabricated metamaterials that are formed by periodic variation of the elastic modulus and mass density of the material. One of the main properties of the phononic crystals is the occurrence of the band-gap of frequency in which phonons (elastic vibration and wave) are not transmitted through the material. For phononic crystals with periodic micro-structures, the origin of the band-gap attributes to two different mechanisms: Bragg scattering and local resonance [1, 2, 3, 4]. However, whether band-gap appears in amorphous materials is still unclear in theory. In this short paper, we show that if the nonlocality is concerned, the band-gap with the scope of $[0, \omega]$ will occur in elastic string without periodic micro-structures.

The origin of nonlocality is regarded due to the long-range interactions within body [5, 6, 7, 8, 9]. Consider a nonlocal elastic string with two clamped ends. The Lagrangian of the nonlocal elastic string takes the following form:

$$L = \frac{1}{2}\rho(\frac{\partial w}{\partial t})^2 - \frac{1}{2}T(\frac{\partial w}{\partial x})^2 - \frac{1}{2}aw\langle w\rangle, \tag{1}$$

where w denotes displacement. ρ and T are the density and tension of the string. a is a coupling constant characterizing the nonlocal effects. $\langle w\rangle$ is the nonlocal term characterizing the long-range interaction within the string, which reads

$$\langle w\rangle = w - \frac{1}{l}\int_0^l w(x)\mathrm{d}x, \tag{2}$$

where l is the length of the string. Inserting Eq.(1) in the nonlocal Lagrangian formulations [10], we have

$$c^2\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial t^2} = \frac{a}{\rho}[w - \frac{1}{l}\int_0^l w(x)\mathrm{d}x], \tag{3}$$

where $c = \sqrt{T/\rho}$, being the wave velocity. Let ω_j and s_j represent the eigenfrequency and wave number of the string, respectively. When $(\omega_j/c)^2 > a/T$, the characteristic equation of Eq.(3) is given as follows:

$$\sin(s_j l) - \frac{2ac^2}{Tls_j\omega_j^2}[1 - \cos(s_j l)] = 0, \tag{4}$$

where ω_j and s_j satisfy the dispersive relation below:

$$s_j^2 = (\frac{\omega_j}{c})^2 - \frac{a}{T}. \tag{5}$$

If $a = 0$, Eq.(5) will revert to the characteristic equation of the ordinary (local) string vibration. So the magnitude of a has direct influences on the nonlocal features of the eigenfrequency. Using Eq.(4) and (5), we calculate the change of ω_1/c, ω_2/c and ω_3/c with a/T, as illustrated in Figure 1 (a). The results show that every order eigenfrequency increases with a rise in a/T, and when a/T approaches 10, the eigenfrequency will jump from the first-order level into the second-order level, the second-order into the third-order, and so on. Therefore, when $a/T > 10$, no first-order mode exists in the vibration of the string. Clearly, it is unrealistic in physics. So a reasonable conclusion is that a/T does not exceed 10. This gives an upper bound of the coupling constant a.

If a concentrated harmonic excitation is prescribed, Eq.(3) can be rewritten as

$$c^2\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial t^2} = \frac{a}{\rho}[w - \frac{1}{l}\int_0^l w(x)\mathrm{d}x] - \frac{P_0}{\rho}\delta(x - \bar{x})\sin(\omega t), \tag{6}$$

where $0 \le \bar{x} \le l$. The steady state solution of Eq.(6) is assumed in the form

$$w(x,t) = f(x)e^{-i\omega t}. \tag{7}$$

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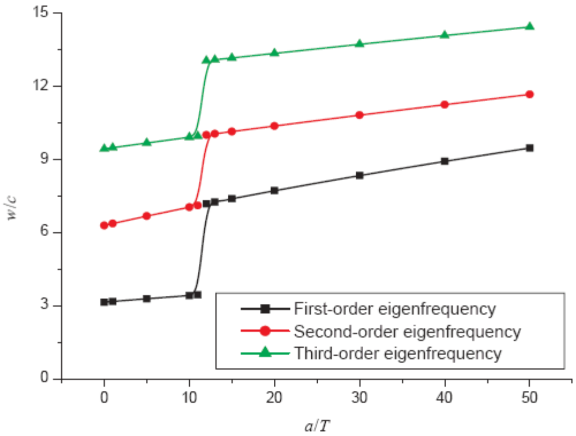

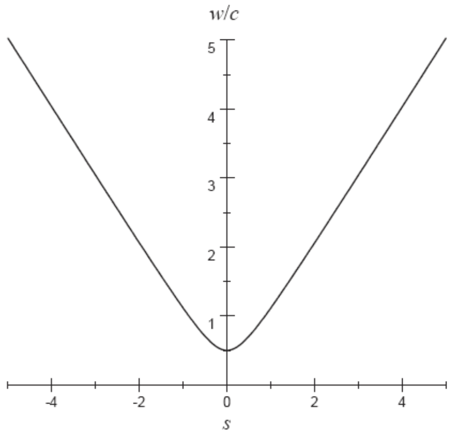

Figure 1. (a) Nonlocal effects of the eigenfrequency (b) Band-gap in $\omega/c - s$ curve ($a/T = 0.25$)

Substituting Eq.(7) into (6) leads to

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + s^2 f = b - \frac{P_0}{T}\delta(x - \bar{x}), \tag{8}$$

where

$$s^2 = (\frac{\omega}{c})^2 - \frac{a}{T}. \tag{9}$$

The general solution of Eq.(8) has the form below:

$$f(x) = \frac{ac^2}{Tls\omega^2}\{q[\cos(sl) - 1] - p\sin(sl)\} + p\cos(sx) + q\sin(sx) - \frac{P_0}{Ts}\int_0^x \delta(\zeta - \bar{x})\sin[s(x - \zeta)]\mathrm{d}\zeta. \tag{10}$$

Using the boundary conditions $f(0) = 0$ and $f(l) = 0$, we have

$$p = \frac{\frac{P_0 ac^2}{lT^2 s^2\omega^2}[1 - \cos(sl)]\sin[s(l - \bar{x})]}{\sin(sl) - \frac{2ac^2}{Tls\omega^2}[1 - \cos(sl)]}, \quad q = \frac{\frac{P_0}{Ts}[1 - \frac{ac^2}{Tls\omega^2}\sin(sl)]\sin[s(l - \bar{x})]}{\sin(sl) - \frac{2ac^2}{Tls\omega^2}[1 - \cos(sl)]}. \tag{11}$$

In terms of Eq.(5) and (9), if $\omega = \omega_j$, we have $s = s_j$. Under this case, it is easy to see that $p \to \infty$ and $q \to \infty$ due to Eq.(4). This result shows that the resonance occurs in the string.
From Eq.(9), we can find an interesting result: only when $\omega \geq c\sqrt{a/T}$ (i.e., $\sqrt{a/\rho}$), can vibration be excited. – This means that there exists a frequency band-gap ranging from 0 to $\sqrt{a/\rho}$. If the frequency of external excitation is in the band-gap, no response will occur. So we can realize an isolation for any low-frequency vibration by controlling the coupling constant a. When $a/T = 0.25$, the change ω/c with s is depicted in Figure 1 (b), which shows that the band-gap of ω/c is $[0, 0.5]$.

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