NONLINEAR RESONANT ULTRASOUND SPECTROSCOPY FOR ONE-DIMENSIONAL HYPERELASTIC SOLID

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Summary The free-vibration acoustic resonance of a one-dimensional hyperelastic solid has been formulated within a framework of the calculus of variation. The strain energy density includes a cubic term of the displacement gradient which is responsible for geometric and constitutive nonlinearities. Numerical analysis based on the Ritz method revealed that resonance frequency of the medium depends on vibration amplitude as well as a nonlinearity parameter. This result indicates that one could determine both the second- and third-order elastic constants from RUS experiments. Prominent nonlinear features, such as symmetry breaking in vibration pattern and small oscillation of pseudo-nodal points, have also been observed.

INTRODUCTION

In 1971, H. H. Demarest demonstrated that the second-order elastic constants C_{ijkl} of a rectangular parallelepiped solid can be determined from inverse analysis of the free vibration resonance frequencies [1]. Several researchers contributed further development of the method and, in consequence, it has been extended to fully anisotropic and piezoelectric materials. Nowadays, this method is called resonant ultrasound spectroscopy (RUS). RUS affords several advantages compared to previous ones: (i) it determines a complete set of elastic constants C_{ijkl} from a frequency sweep to *one* single crystal specimen, (ii) it is applicable to mm-ordered small-sized specimen and (iii) measurement accuracy is sufficiently high; a typical error is less than $\sim 1\%$. However, the theory of RUS is still in development and further improvements are required. The most promising one would be extending the theory into a general framework of nonlinear elasticity. Since the pioneering work done by Demarest, linear theory of elasticity has been employed. This approximation simplifies the RUS theory on one hand, however, it imposes a strict constraint on the other: we fail to determine nonlinearity, or so-called the higher-order elastic constants, from RUS experiment. The aim of this study is to formulate free vibration acoustic resonance of a nonlinear hyperelastic solid. Our theory enables one to determine both the second- and third-order elastic constants, thereby serves as nonlinear RUS (NRUS).

MATHEMATICAL PRELIMINARIES

To simplify the analysis we employ the one-dimensional hyperelastic bar $\Omega = \{x \in \mathbb{R} \,|\, x \in [-1, 1]\}$ which is assumed to be uniform in the reference configuration. The bar is subjected to a pure longitudinal displacement u, which is a function of position $x \in \Omega$ and time $-\infty < t < \infty$; $u = u(x, t)$. Let ρ, $u_t = \partial u/\partial t$ and $u_x = \partial u/\partial x$ be the mass-density, displacement velocity and displacement gradient defined in the reference state. The kinetic energy density can be written as $\mathcal{T} = \rho u_t^2/2$. On the other hand, the strain energy density is defined by $\mathcal{W} = k_1 u_x^2/2 - k_2 u_x^3/6 + o(u_x^3)$, where $k_1 = C_{1111}$ and $k_2 = -(3C_{1111} + C_{111111})$. Note that the cubic term is responsible for both geometric and constitutive nonlinearities [2, 3]. The lagrangian density is expressed by $\mathcal{L} = \mathcal{T} - \mathcal{W}$. Let ω be a resonance frequency of the bar. Integration of $\mathcal{L}$ with respect to the domain Ω and time interval $t = [-\pi/\omega, \pi/\omega]$ yields the action integral such that $I[u] = \int_{-\pi/\omega}^{\pi/\omega} \int_\Omega \mathcal{L}(u_t, u_x)\, dxdt$. Since the bar Ω is a holonomic system and free from any dissipatioin, the principle of stationary action (Hamilton's principle) should hold; the displacement function u must satisfy the stationary condition $\delta I = 0$. In order to solve the variable domain variational problem we consider the following transformations: $x \to x$, $t \to t + \epsilon\psi^t(x, t, u_i, u_{i,j}, u_t) + o(\epsilon)$ and $u \to u + \epsilon\psi^u(x, t, u_i, u_{i,j}, u_t) + o(\epsilon)$. Accordingly, the first variation of the functional becomes

$$\delta I = \int_{t_1}^{t_2} \int_\Omega \Big(\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{u_t} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x} \Big) \bar{\psi}^u dxdt + \int_{t_1}^{t_2} \int_\Omega \Big[\frac{\partial}{\partial t}\Big(\frac{\partial \mathcal{L}}{\partial u_t}\bar{\psi}^u + \mathcal{L}\psi^t \Big) + \frac{\partial}{\partial x}\Big(\frac{\partial \mathcal{L}}{\partial u_x}\bar{\psi}^u \Big) \Big] dxdt.$$

Where $\bar{\Psi}^u = \Psi^u - u_t\Psi^t$. The second term in right-hand side (R.H.S.) of the equation represents the boundary conditions. In this study, we impose natural boundary conditions for t and x. In consequence, we obtain the time periodicity condition $u(x, t - \pi/\omega) = u(x, t + \pi/\omega)$ for all $x \in \Omega$ and stress-free condition for all t on the edges [2, 3]. The first term in R.H.S. represents the weak form of Euler-Lagrange equation. Unfortunately, no analytic solution has been reported in its local form, $\rho u_{tt} - (k_1 - k_2 u_x)u_{xx} = 0$, and hence, we solve the variational problem directly using the Ritz method. Here we employed the complex Fourier series for the basis function and expanded the displacement function u in such a way that

$$u = \sum_{n=-N}^{N} \Big[\frac{\alpha_{n,0}}{2} + \sum_{m=1}^{M} \Big(\alpha_{n,m} \cos\frac{m\pi x}{L} + \beta_{n,m} \sin\frac{(2m-1)\pi x}{2L} \Big) \Big] e^{in\omega t}.$$

Inserting the u into the functional $I[u]$ and imposing the stationary condition $\delta I = 0$, it ends up with a nonlinear eigenvalue problem. It has been solved numerically through a convergence calculation by the Newton method.

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RESULTS AND DISCUSSION

Numerical analysis revealed that the hyperelastic bar Ω shows prominent nonlinear features. The most important one is that the resonance frequency ω depends on the vibration amplitude and the nonlinearity parameter k_2/k_1. Figure 1 (a) shows the amplitude dependence of the resonance frequency obtained from the fourth resonant vibration mode. Here, four plots show the four cases of k_2/k_1. As seen from the figure, resonance frequencies at the lowest amplitude is $\omega = 2\pi$ and which is irrespective to the value of k_2/k_1. This result is reasonable since $\omega = 2\pi$ coincides with the linear case solution and since the present model is equivalent to the linear one when the displacement, and therefore the displacement gradient u_x, is infinitesimally small. On the other hand, ω decreases monotonically with increasing the amplitude, where the slope depends significantly on the nonlinearity parameter k_2/k_1. This result has crucial importance because it ensures that one could determine the parameters k_1 and k_2, and therefore the second- and third-order elastic constants, from the amplitude dependence of resonance frequency ω. Note that, to our surprise, the amplitude dependence of resonance frequency is similar to that of the temperature dependence obtained by RUS experiments. Another issue which we would like to address here is that resonant vibration pattern is slightly distorted from the corresponding linear one. Figure 1(b) shows the resonant vibration pattern obtained from the fourth mode at $k_2/k_1 = 1$ and $\|u\|_{L^2} = 0.08$. The corresponding resonance frequency is $\omega = 6.15$. In this figure, we can observe prominent features due to the nonlinearity of the system. First of all, the vibration pattern is no longer symmetric with respect to time. For instance, the displacement u at $t = 0$ on the edge $x = 1.0$ is $|u| = 1.33$ while that after a half period ($t = \pi/\omega$) is only $|u| = 0.7$. In addition, there is no mirror symmetry on the x-axis for all t. Secondly, thre is no specific time t^* in which displacement vanishes $u(x, t^*) = 0$ for all $x \in \Omega$. Finally, there is no vibration node except at $x = 0$. For the case of linear system there are four nodal points: $x = \pm 0.25$ and $x = \pm 0.75$. However, as seen from Fig. 1(b), they are slightly oscillating throughout the resonant vibration. Hence, except for $x = 0$, there is no nodal point x^* at which displacement vanishes $u(x^*, t) = 0$ for all t.

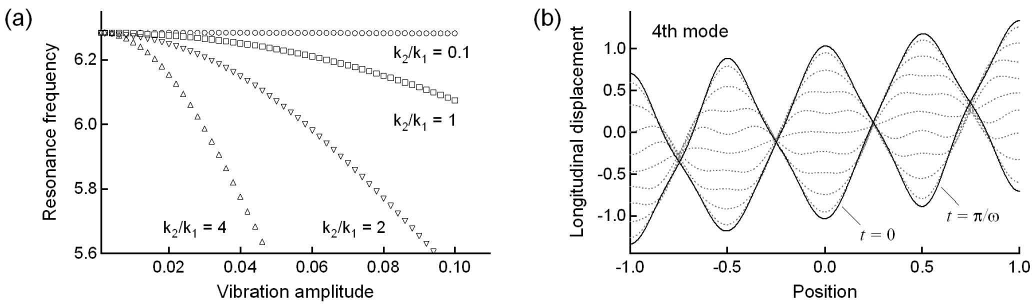

Figure 1. (a) amplitude dependence of resonance frequency of the fourth mode and (b) resonant vibration pattern obtained from the fourth mode at $k_2/k_1 = 1$ and $\|u\|_{L^2} = 0.08$. Note that the vibration amplitude is defined by the L^2 norm of u at $t = 0$: $\|u\|_{L^2} = (\int_\Omega u^2(x, 0)dx)^{1/2}$.

CONCLUSIONS

In this study, we have formulated free vibration acoustic resonance of a nonlinear hyperelastic bar Ω within a framework of the calculus of variation. The bar Ω is subjected to a pure longitudinal displacement $u = u(x, t)$ and the resonant state is derived from a stationary condition to the action integral. Numerical analysis based on the Ritz method revealed that the resonance frequency depends on the vibration amplitude as well as the nonlinearity parameter k_2/k_1. This result is crucially important since it ensures that we could determine both the second- and third-order elastic constants from RUS experiments. We also observed the symmetry breaking and loss of nodal point in the resonant vibration pattern.

References

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