COMPARISION OF NON-EUCLIDEAN CONTINUUM MODEL WITH STRAIN GRADIENT THEORY

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Summary The non-Euclidean theory is compared with the strain gradient theory for the purpose of describing the stress and strain distribution. It is shown that both models give the same result for the irrotational displacement field and the stress field. The proposed analysis allows us to link the non-Euclidean characteristics with the Eulerian strain.

INTRODUCTION

In classical continuum theories the material metric G_{ij} is always Euclidean, or can be reduced to a Euclidean one by a coordinate transformation. However if this metric is related to the local atomic structure then this cannot be possible. The physical interpretation is that an internal disarrangement of the atomic structure of a material neighborhood will cause its metric to become "intrinsically" non-Euclidean one g_{ij} .

In the physical literature even in the middle 1950s Kondo and Bilby [1, 2] pointed out the need for introduction of the non-Euclidean characteristics. They tried to determine a conceptual connection between the micromotion and the global material behavior. It was shown that introduction of "comparison crystal" for description of linear dislocations leads to the appearance of asymmetric connection in the object theory forbidden by Euclidean geometric structure in the classical theory. It means that the internal geometry of a material doesn’t coincide with the geometry of the observer's Euclidean space. The variants of generalization of elasticity classical theory within the framework of various non-Euclidean models of continuous medium were proposed by many authors, e.g., [3-7].

From the physical point of view the geometric characteristics of the affine-metric spaces are the internal variables, and they can't be measured directly. Therefore, verification of the results of non-Euclidean theory can be based on a comparison with results of other approaches. For this purpose we use the strain gradient theory formulated by Toupin and Mindlin. In early 1960s, Toupin [8] and Mindlin [9] proposed a theory in which the strain energy function is assumed to depend on both the strain and strain gradient. Numerous extensions of this theory have since been developed and used for various applications to investigate such problems as strain localization and size effects in materials and challenging issues on the micro/nano scales. It is noteworthy that there are many other gradient theories that have also received much attention from many engineering fields. In this paper, however, we do not attempt to make a comprehensive comparison among the various gradient theories, for which purpose the readers are referred to more recent papers [10].

MAIN ASSUMPTIONS

According to the general idea of the Toupin–Mindlin theory it is assumed that the bulk the Helmholtz free energy Ψ_{TM} depends on Eulerian strain $A_{ij}=(\delta_{ij}-G_{ij})/2\approx\left(\partial U^i/\partial x^j+\partial U^j/\partial x^i\right)/2$ and its gradients. For simplicity of the further analysis we assume that the free energy is equal to $\Psi_{TM}=\Psi_{TM}(A_{ij},\nabla A)$ where $A=A_{kk}$ is the first strain invariant. For small strains the free energy is written in the form $\Psi_{TM}=\lambda A^2/2+\mu A_{ij}A_{ij}+G\left|\nabla A\right|^2/2$ where λ,μ are conventional Lamer constants, and G is an elastic constant associated with gradient term.

In the non-Euclidean theory the set of additional parameters $\varepsilon_{ij}=(\delta_{ij}-g_{ij})/2$ is introduced to describe the mechanical state of a body containing defects. Structure of the components ε_{ij} is different. The functions A_{ij} satisfy the Saint-Venant compatibility conditions which violate for ε_{ij}. In the case of the plane strain approximation there is a unique function

$$R=\frac{\partial^2\varepsilon_{11}}{\partial x^2\partial x^2}-2\frac{\partial^2\varepsilon_{12}}{\partial x^1\partial x^2}+\frac{\partial^2\varepsilon_{22}}{\partial x^1\partial x^1}$$

characterizing incompatibility of the components ε_{ij}. It is shown in works in the theory of elasticity that the vanishing of R in a region is necessary and sufficient condition for the equality $\varepsilon_{ij}=A_{ij}$. From the mathematical point of view R is the scalar curvature of the Riemann tensor R_{ijkl} . If $R_{ijkl}=0$ then the internal geometry of a body is the Euclidean one. In the case of $R_{ijkl}\neq 0$ we have a non-Euclidean internal geometry of the body. For the plane strain approximation the scalar curvature is the unique invariant of the Riemann tensor. Then the set of kinematic variables for the non-Euclidean Riemann continuum model includes $A_{ij}, \varepsilon_{ij}, R$ and the bulk Helmholtz free energy is equal to $\Psi_{NE} = \Psi_{NE}(A_{ij}, \varepsilon_{ij}, R)$. We write the bulk free energy in the form: $\Psi_{NE} = \Psi_A + \Psi_\varepsilon + \Psi_{A\varepsilon} + \Psi_R$. Here Ψ_A and Ψ_ε depend on A_{ij} and ε_{ij} respectively, and $\Psi_{A\varepsilon}$ characterizes the interaction of the fields A_{ij} and ε_{ij}; Ψ_R is a function of a scalar parameter R. Since we consider small strains then the functions Ψ_A and Ψ_ε are assumed to depend on first and second invariants of their tensor arguments, and $\Psi_{A\varepsilon}$ depends on product of $A_{nn}\varepsilon_{kk}$.

MAIN RESULTS

Constitutive equations of the strain gradient theory and the non-Euclidean theory are obtained from the condition that the deformation of the body minimizes the corresponding total energies $\int \Psi_{TM} dV$ and $\int \Psi_{NE} dV$. We analyzed the structure of the irrotational displacement field $\mathbf{U}$ and the stress field σ_{ij} for both models.

It is shown that the displacement field has the following form $\mathbf{U} = \mathbf{U}_{\mathrm{cl}} + \nabla\Phi$ where $\mathbf{U}_{\mathrm{cl}}$ coincides with the elastic stain field. The function Φ defines the scalar curvature: $\Phi = \alpha R$. It satisfies the equation $\Delta\Phi + \Gamma\Phi = 0$ with phenomenological parameters α, Γ and the Laplace operator Δ . Calculation of the stress field results in $\sigma_{ij} = \sigma_{H,ij} + T_{ij}$ where $\sigma_{H,ij}$ coincides with the elastic stress field determined in accordance with Hooke's law. An additional fields $T_{ij} = \beta\left(\delta_{ij}\Delta R - \partial^2 R / \partial x^i \partial x^j\right)$ are the solutions of the equilibrium equations. They are also self-equilibrated fields, i.e. they satisfy the integral conditions of mechanical equilibrium for an arbitrary volume inside the body.

Hence the non-Euclidean theory and the strain gradient theory define the inhomogeneous structure of the displacement and stress field different from the classical one. The proposed analysis allows us to link the non-Euclidean characteristics with the Eulerian strain. In the previous work [11] the non-Euclidean theory was used for the description of the stress-field distribution around underground excavations.

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