THREE-DIMENSIONAL THERMAL-STRESS ANALYSIS OF NON-HOMOGENEOUS ELASTIC COMPOSITES

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Summary This talk addresses an efficient approach for three-dimensional analysis of elastic and thermoelastic response of a non-homogeneous composite layer to external force and thermal loadings. The main feature of this approach consists in integration of the equilibrium equations, which do not depend on the material properties. The original elasticity and thermoelasticity problems are reduced to solution of the Volterra integral equations of second kind for each stress-tensor component. By making use of the resolvent-kernel technique, the solution is suggested in an explicit form for arbitrary dependences of the material properties on the thickness coordinate.

RESEARCH OBJECTIVES AND MOTIVATION

Elastic and thermoelastic analyses are quite important for adequate evaluation of operational performance of engineering structures or machinery operating under action of significant thermal and/or mechanical loadings in hostile environments, and, thus, they present a great deal of interest for scientists in both academia and industry. Advanced mathematical modeling and efficient methodology for providing such analyses require a wide range of the material parameters to be taken into consideration. Among a number of the effective parameters, the non-homogeneity (material properties depend on the spatial coordinates) and thermo-sensitivity (material properties are functions of the non-uniform temperature) along with the general anisotropy received a considerable attention in recent decades. Particularly, the growing interest to the materials with abovementioned properties is caused by the widespread application of the functionally graded materials [1], whose properties can be formed technologically to achieve the required operational performance of the structure members. However, accounting of the material non-homogeneity brings considerable difficulties into analytical treatment of relevant problems of elasticity and thermoelasticity and influences substantially the accuracy [2]. The main difficulty lies in the need to solve the partial differential equations with variable coefficients. This fact makes it impossible, except for very few particular cases, to solve the problems analytically. In the existing literature, this difficulty has been obviated in several ways concerning appropriate simplifications, such as application of approximate solution methods or linearization techniques; development of analytical methods for the materials with preliminary given material properties (in the form of linear, power, or exponential functions, etc.) those present no difficulty for construction of solutions to the governing equations; representation of a continuously inhomogeneous solid by a composite consisting of a number of tailored homogeneous layers, etc.

Despite a variety of the abovementioned methods, there is a strong need in analytical or semi-analytical methods applicable for rather general functional forms of the material properties. Moreover, the efficient analysis of the problems on proper evaluation and optimization of the operational performance, optimal heating control, etc., requires the solutions of relevant elasticity and thermoelasticity problems in a most convenient explicit form.

This talk addresses an approach for analysis of the three-dimensional elasticity and thermoelasticity problems for a non-homogeneous layer subjected to external force and thermal loadings. This approach has been efficiently employed for the analysis of one- and two-dimensional problems [3, 4]. The main feature of our approach consists in direct integration of the equilibrium equations, which are in terms of stresses, and, thus, do not involve the material properties. This fact allows for expression of all the stress-tensor components in terms of the governing functions (the total stress, and some of the normal stress-tensor components), as well as derivation of the integral equilibrium conditions for the governing functions on the basis of the original boundary conditions. By making use of the obtained relations, the compatibility equations with arbitrarily-variable coefficients can be reduced to the appropriate equations for the governing functions. In such manner, the original problems are reduced to finding the governing functions. By reducing the governing equations to the Volterra integral equations of second kind, we solve them by means of the resolvent-kernel technique.

FORMULATION OF THE PROBLEM AND THE SOLUTION TECHNIQUE

We consider the three-dimensional thermoelasticity problem for a non-homogeneous composite layer $\{ |x| < \infty, |y| < \infty, |z| \le h \}$, whose material properties are arbitrary functions of the thickness-coordinate z. The problem is governed by the classical equilibrium equations, constitutive strain-stress equations, and the Cauchy's strain-displacement relations [5]. The layer is subjected to the action of a non-uniform temperature $T = T(x, y, z)$ and external force loadings

$$\sigma_z(x,y,\pm h) = -p_z^{\pm}(x,y), \quad \gamma_{xz}(x,y,\pm h) = q_{xz}^{\pm}(x,y), \quad \gamma_{yzy}(x,y,\pm h) = q_{yz}^{\pm}(x,y)$$

on its limiting planes $z = \pm h$. Here σ_z, τ_{xz}, and τ_{yz} are the normal and shearing stress-tensor components related to *z*-direction, $p_z^{\pm}, q_{xz}^{\pm}, q_{yz}^{\pm}$ are given functions, temperature T can be determined from the steady-state heat transport equation

$$\kappa(z)\frac{\partial^2 T}{\partial x^2} + \kappa(z)\frac{\partial^2 T}{\partial y^2} + \frac{\partial}{\partial z}\left(\kappa(z)\frac{\partial T}{\partial z}\right) + Q = 0$$

with appropriate boundary conditions [6], where κ denotes the thermal conductivity and Q is the density of internal heat sources.

The considered problem formulation presents 16 interdependent equations for 16 quested-for functions (6 stresses, 6 strains, 3 displacements, and the temperature). To solve this problem for the non-homogeneous layer, we adopt the technique of direct integration [7] proposed for the case of homogeneous material. By making use of this technique, the original problem can be reduced to the following set of governing equations:

$$\begin{aligned} &\Delta\sigma_z = \frac{1}{1+\nu}\Delta_{xy}\left(\sigma + \alpha E T\right), && \Delta\left(\frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}\sigma + \alpha T\right) = 2\frac{\partial \tau_{xz}}{\partial x}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1+\nu}{E}\right) - \frac{1}{E}\frac{\partial^2\sigma}{\partial x^2} - \alpha\frac{\partial^2 T}{\partial x^2}, \\ &\Delta\left(\frac{1-\nu}{E}\sigma + 2\alpha T\right) = \sigma_z\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{1+\nu}{E}\right), && \Delta\left(\frac{1+\nu}{E}\sigma_y - \frac{\nu}{E}\sigma + \alpha T\right) = 2\frac{\partial \tau_{yz}}{\partial y}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1+\nu}{E}\right) - \frac{1}{E}\frac{\partial^2\sigma}{\partial y^2} - \alpha\frac{\partial^2 T}{\partial y^2} \end{aligned} \tag{1}$$

with appropriate boundary and integral conditions, those are the same as for the homogeneous case [7]. Here $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplace operator, $\Delta_{xy} = \Delta - \partial^2/\partial z^2$, $\sigma = \sigma_x + \sigma_y + \sigma_z$ is the total stress, $E(z)$ stands for the Young's modulus, $\nu(z)$ is the Poisson's ratio, $\alpha(z)$ denotes the linear thermal expansion coefficient.

By making use of the Fourier transformation with respect to the planar coordinates *x* and *y*, we can separate the variables in equations (1). Having determined the stress σ_z from the first equation of (1) and then substituted it into the third equation, we can obtain the Volterra integral equation for the total stress σ only. Analogously, the integral equation for σ_z can be derived. These equations can be solved explicitly be means of the resolvent-kernel technique [3,4]. Once the total stress is determined, it can be substituted into the second and fourth equations of (1) along with expressions for the shearing stresses trough the normal ones [7] obtained by means of direct integration of the equilibrium equations. In such manner, we can achieve the integral Volterra equations for the stresses σ_x and σ_y, which can be solved by means of the resolvent-kernel technique. Then the shearing stresses can be found from the relations obtained on the basis of the equilibrium equations. After all the stress-tensor components are constructed as the Fourier transformations, the originals of stresses can be reproduced by making use of the inverse Fourier transformation.

CONCLUDING REMARKS

We propose a method for solution of the three-dimensional elasticity and thermoelasticity problems for a non-homogeneous composite layer subjected to the thermal and force loadings. Being based upon the technique of direct integration of the equilibrium equations, this method allows for reducing the original problems to solution of the Volterra integral equations of second kind formulated for the separate stress-tensor components. Our method shows the following advantages:

- *i)* the solution is suggested in an explicit functional form;
- *ii)* the resolvent-kernel is expressed through the material properties only and it does not depend on the external or thermal loading;
- *iii)* the method can be applied for material properties of arbitrary functional form.

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