ON THE THEORY OF ELASTICITY WITH RESIDUAL SURFACE STRESSES WITH APPLICATIONS AT THE MICRO- AND NANOSCALE

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Summary Using the Gurtin–Murdoch model of surface stresses [1] we consider the influence of residual (initial) surface stresses on the effective (apparent) properties of elastic solids such as the effective Young modulus of nano-porous rods or the bending stiffness of plates and shells. Within the framework of this model the elastic body can be considered as a non-linear elastic 3D body with an attached 2D elastic membrane. We consider the case when the bulk material and the attached elastic membrane have different reference placements. We assume that the reference placement for bulk material is the natural one, that is the stresses are vanished when the corresponding strains are vanished. For the attached membrane we assume the non-natural reference placement for which surface stresses exist in the case of zero surface strains. We analyze how the residual stresses influence on tension of porous rods and bending of plates and shells.

INTRODUCTION

The recent progress in nanotechnologies is based on the production of new materials, so-called nanomaterials, whose properties can differ significantly from the properties of bulk materials. One of the explanations for these differences consists of the presence of surface effects, whose role can be extremely large for nano-dimensional structures in comparison with those in the classical elasticity, see [2, 3, 4]. Since the contributions of Laplace, Young, and Gibbs the mathematical study of boundary-value problems (BVP) for elastic bodies with surface stresses have been carried out in many works, see e.g. [2, 3, 4] and references therein. A mathematical study of the BVP of linear elasticity with surface stresses is given in [5, 6]. Recently, the theory is applied to formulation of theories of nano-sized structures, see e.g. [7, 8].

EQUILIBRIUM OF AN ELASTIC BODY WITH SURFACE STRESSES

Following [1] we consider the finite deformations of the elastic solid with surface stresses. The equilibrium equations and the boundary conditions of the elastic body with surface stresses are given by

$$\nabla_{\mathbf{x}} \cdot \mathbf{P} + \rho \mathbf{f} = \mathbf{0}, \quad (\mathbf{N}\cdot\mathbf{P} - \nabla_S \cdot \mathbf{S})\big|_{\Omega_s} = \mathbf{t}, \qquad \mathbf{u}\big|_{\Omega_u} = \mathbf{0}, \quad \mathbf{N}\cdot\mathbf{P}\big|_{\Omega_f} = \mathbf{t}. \tag{1}$$

Here $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, $\mathbf{S}$ is the first Piola-Kirchhoff type surface stress tensor acting on the surfaces Ω_s, $\nabla_{\mathbf{x}}$ and ∇_S are the 3D and 2D nabla operators in the reference configuration, respectively, $\mathbf{N}$ is the unit normal to $\Omega \equiv \Omega_u \cup \Omega_f \cup \Omega_s$, $\mathbf{u} = \mathbf{x} - \mathbf{X}$ is the displacement vector, $\mathbf{x}$ and $\mathbf{X}$ are the position vectors in the actual and reference configurations, respectively, $\mathbf{f}$ and $\mathbf{t}$ are the body force and surface load vectors, respectively, and ρ is the density in the reference configuration. We assume that the part of the body surface Ω_u is fixed, while on Ω_f the surface stresses are absent.

For $\mathbf{P}$ we use the standard constitutive equation of the nonlinear elasticity, while the constitutive equation for $\mathbf{S}$ is similar to the membrane stress resultant

$$\mathbf{P} = \frac{\partial \mathcal{W}}{\partial \nabla_{\mathbf{x}} \mathbf{x}}, \quad \mathbf{S} = \frac{\partial \mathcal{U}}{\partial \mathbf{F}},$$

where $\mathcal{W} = \mathcal{W}(\nabla_{\mathbf{x}}\mathbf{x})$ is the strain energy density function, $\mathcal{U} = \mathcal{U}(\mathbf{F})$ the surface strain energy density, $\mathbf{F} = \nabla_S \mathbf{x}_S$ is the surface deformation gradient defined on Ω_s and $\mathbf{x}_S = \mathbf{x}\big|_{\mathbf{x}\in\Omega_s}$ is the position vector of Ω_s.

In what follows we assume that W and $\mathbf{P}$ possess the properties $\mathcal{W}(\mathbf{I}) = 0, \quad \mathbf{P}(\mathbf{I}) = \mathbf{0}$, while there exist residual (initial) surface energy and surface stresses that is

$$\mathcal{W}(\mathbf{A}) = \mathcal{W}_0 \neq 0, \quad \mathbf{S}(\mathbf{A}) = \mathbf{S}_0 \neq \mathbf{0},$$

where $\mathbf{I}$ and $\mathbf{A} \equiv \mathbf{I} - \mathbf{N}\otimes\mathbf{N}$ are the 3D and surface unit tensors, respectively. In other words we assume that the reference placement for bulk material is natural one while for the attached on Ω_s membrane we assume the non-natural reference placement. Further we consider the influence of W_0 and $\mathbf{S}_0$ on the effective (apparent) properties of solids.

TENSION OF A NANO-POROUS ROD

To illustrate the more complex model we consider the simple one-dimensional (1D) case, that is the problem of the uniaxial stress state of a circular rod of radius R with n identical channels of radius r taking into account surface effects.

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The channels are uniformly distributed parallel to the rod axis. We introduce the porosity ϕ by the relation $\phi = S/F$, $\phi \in [0, 1)$, where F is the area of the rod cross-section and S is the total area of channel's cross-sections in the reference placement. The tensile force P^* depends on the longitudinal stretch λ as follows

$$P^* = P(\lambda) + P_S(\lambda). \tag{2}$$

Here P and P_S are determined by the bulk and surface material properties, respectively. Defining the effective Young modulus of porous rod by the formula $E^* = \frac{\mathrm{d}}{\mathrm{d}\lambda} P^*\big|_{\lambda=1}$ we extend [9] and derive the following formula

$$E^* = E(1-\phi) + E_S \frac{2\sqrt{\phi F}}{\sqrt{\pi}} \sqrt{n}, \tag{3}$$

where E is the Young modulus of the bulk material and E_S is the tangential surface elastic modulus. Let us note that E_S now depends on the residual stretch or on the residual stresses and in general can take positive or negative values. The negative value of E_S corresponds to the loss of stability of the attached elastic membrane, see [10] as an example of bifurcation analysis of solids with surface elasticity. Thus Eq. (3) shows that the porous rod can be stiffer or softer than the porous rod without surface stresses.

BENDING OF NANO-SIZED PLATES AND SHELLS

Applying the through-the-thickness integration technique of [11] to shell-like bodies with surface stresses, we obtain the 2D constitutive equations for nano-sized plates and shells in the following form, see [7],

$$\mathbf{T}^* = \mathbf{T} + \mathbf{T}_S\,, \quad \mathbf{M}^* = \mathbf{M} + \mathbf{M}_S\,, \tag{4}$$

$$\mathbf{T} = \int \mathbf{G}\cdot\mathbf{P}\, d\zeta\,, \quad \mathbf{M} = -\int \mathbf{G}\cdot\mathbf{P}\times\mathbf{z}\, d\zeta\,, \quad \int(\ldots)\, d\zeta = \int_{h_-}^{h_+} (\ldots)\, d\zeta\,, \tag{5}$$

$$\mathbf{T}_S = G_+\nabla_S\cdot\mathbf{S}_+ + G_-\nabla_S\cdot\mathbf{S}_- \quad \mathbf{M}_S = -\frac{h}{2}\left[G_+(\nabla_S\cdot\mathbf{S}_+)\times\mathbf{z}_+ - G_-(\nabla_S\cdot\mathbf{S}_-)\times\mathbf{z}_-\right], \quad G_\pm = G\big|_{\zeta=\pm h/2}. \tag{6}$$

where $\mathbf{T}$ and $\mathbf{M}$ are the first Piola-Kirchhoff stress and couple stress resultant tensors, respectively. Equations (4) are the 2D analogue of (2). $\mathbf{T}$ and $\mathbf{M}$ are the classical tensors [11], while $\mathbf{T}_S$ and $\mathbf{M}_S$ are resultant tensors induced by surface stresses $\mathbf{S}_\pm$ acting on the shell faces [7]. In addition, $\mathbf{z}$ is the base reference deviation and $\mathbf{G} \equiv -\mathbf{N}\times(\mathbf{A} - \zeta\nabla_S\mathbf{N})\times\mathbf{N}$ is the geometrical tensor, $G \equiv \det(\mathbf{A} - \zeta\nabla_S\mathbf{N})$ is the geometric scale factor defined in [11], ζ is the coordinate along the unit normal $\mathbf{N}$ in the reference placement, and h is the shell thickness. Unlike [8] here $\mathbf{T}_S \neq \mathbf{0}$ and $\mathbf{M}_S \neq \mathbf{0}$ when $\mathbf{u} = \mathbf{0}$ due to residual stresses. These terms may play more important role than $\mathbf{T}$ and $\mathbf{M}$. Obviously, that this influence is significant when the surface effects play a dominant role, i.e. in the case of micro- and nanoscales. In particular, this affects the effective material properties and may lead to self-instability of thin-walled structures at the nanoscale.

CONCLUSIONS

We discuss here the influence of surface effects on the effective properties of materials such as the effective bending stiffness of plates or the stiffness of the rods. We consider the model of surface effects based on the concept of surface stresses which are the generalization of the surface tension for solids. In particular, the residual surface stresses play an important role for such nano-sized materials as films, nano-porous materials, etc. Within the framework of the developed model we present the effective stiffness properties of plates, shells and nano-porous rods. In the linear case the surface elasticity reduces to the addition of new terms to the elastic stiffness parameters. We have shown that the residual surface stresses make a shell or a rod more stiffer or softer in comparison with elements without surface stresses.

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