NUMERICAL SOLUTION OF NON-SMOOTH MULTIBODY SYSTEMS

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1 Abstract

The theory of non-smooth multibody dynamics with unilateral contacts is now well established, for example in terms of measure equations of motion added by complementarities or formulated with the help of differential inclusions. Most researchers today focus on numerical methods for solving these systems, because computing times especially for large systems are a problem. Time-stepping schemes for time-integration, pivoting or iterative algorithms for solving the complementarity problem and the Augmented Lagrangian approach are methods of increasing numerical efficiency for large systems. The paper describes new findings for unilateral multibody systems and presents an industrial roller coaster example.

2 Mathematical Model

A description, which is adequate to the physics of nonsmooth systems, was given by Moreau [4] and in the following years more elaborated by Glocker [3] (see also [5]). Moreau suggested measure differential equations representing the equations of motion for systems with non-continuous phenomena. These equations contain a Lebesgue-measurable part for the continuous components and an atomic part based on a Dirac point measure for the impact parts. They write with $\boldsymbol{u} = \dot{\boldsymbol{q}}$

$$\boldsymbol{M}(\boldsymbol{q},t)d\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{u},\boldsymbol{q},t)dt - \boldsymbol{W}_N(\boldsymbol{q},t)d\boldsymbol{\Lambda}_N - \boldsymbol{W}_T(\boldsymbol{q},t)d\boldsymbol{\Lambda}_T = 0\,. \tag{1}$$

with $\boldsymbol{M}(\boldsymbol{q},t)$ the mass matrix, $\boldsymbol{W}_N, \boldsymbol{W}_T$ the constraint matrices, $\boldsymbol{\Lambda}_N, \boldsymbol{\Lambda}_T$ the constraint impulses. The measure for the velocities $d\boldsymbol{u} = \dot{\boldsymbol{u}}dt + (\boldsymbol{u}^+ - \boldsymbol{u}^-)d\eta$ is split in a Lebesgue-measurable part $\dot{\boldsymbol{u}}dt$, which is continuous, and the atomic parts which occur at the discontinuity points with the left and right limits $\boldsymbol{u}^+$ and $\boldsymbol{u}^-$ and the Dirac point measure $d\eta$. Similarly, the measure for the impulses is defined as $d\boldsymbol{\Lambda} = \boldsymbol{\lambda}dt + \boldsymbol{\Lambda}d\eta$. In addition we apply the proximal point idea from convex analysis to develop the so-called “Augmented Lagrange Method” [1].

In the case of complementarity in normal direction (contact/detachment-case) the set is just the half space with non-negative constraint forces λ_N. Therefore we get the relations

$$\boldsymbol{\lambda}_N = \mathrm{prox}_{C_N}(\boldsymbol{\lambda}_N - r\boldsymbol{g}_N)\,, \qquad C_N(\lambda_N) = \{\boldsymbol{\lambda}_N | \boldsymbol{\lambda}_N \geq 0\}\,. \tag{2}$$

where C_N denotes the set of admissible normal contact forces. In the same manner we can put Coulomb's friction law into the form

$$\boldsymbol{\lambda}_T = \mathrm{prox}_{C_T(\lambda_N)}(\boldsymbol{\lambda}_T - r\dot{\boldsymbol{g}}_T)\,, \qquad C_T(\lambda_N) = \{\boldsymbol{\lambda}_T |\, |\boldsymbol{\lambda}_{Ti}| \leq \mu_{0i}\lambda_{Ni}\}\,. \tag{3}$$

with C_T denoting the set of admissible tangential forces. Combining the equations (1) to (3) we finally obtain a set of non-smooth continuous equations for the unknowns $\ddot{\boldsymbol{q}}$, $\boldsymbol{\lambda}_N$ and $\boldsymbol{\lambda}_T$

$$\begin{aligned}
\boldsymbol{M}(\boldsymbol{q},t)d\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{u},\boldsymbol{q},t)dt - -\boldsymbol{W}_N(\boldsymbol{q},t)d\boldsymbol{\Lambda}_N - \boldsymbol{W}_T(\boldsymbol{q},t)d\boldsymbol{\Lambda}_T &= 0,\\
\boldsymbol{\lambda}_N - \mathrm{prox}_{C_N}(\boldsymbol{\lambda}_N - r\ddot{\boldsymbol{g}}_N) &= 0,\\
\boldsymbol{\lambda}_T - \mathrm{prox}_{C_T(\lambda_N)}(\boldsymbol{\lambda}_T - r\ddot{\boldsymbol{g}}_T) &= 0\,,
\end{aligned} \tag{4}$$

which is a convenient form for applying the time stepping algorithm together with the Augmented Lagrange idea for a numerical solution. An iteration scheme for the r-factors we shall present in the following section.

3 Numerical Model

We discretize (4) in the following form:

$$M\Delta u + h\Delta \mathrm{t} - [(W_N + W_R) \quad W_T]\begin{pmatrix}\Lambda_N\\ \Lambda_T\end{pmatrix} = 0,$$

$$\begin{pmatrix}
\Delta q = q^{l+1} - q^l = (u^l + \Delta u)\Delta \mathrm{t},\\
\dot{g}_N = \quad W_N^T(u^l + \Delta u) + \bar{w}_N\Delta \mathrm{t},\\
\dot{g}_T = \quad W_T^T(u^l + \Delta u) + \bar{w}_T\Delta \mathrm{t},\\
\Lambda_N - \mathrm{prox}_{C_N}(\Lambda_N - r\dot{g}_N) = 0,\\
\Lambda_T - \mathrm{prox}_{C_T(\Lambda_N)}(\Lambda_T - r\dot{g}_T) = 0,
\end{pmatrix}_{l+1} \tag{5}$$

As before the constraints and the relative velocities are evaluated at the time (l+1). The numerical solution of these equations have to start with a choice of the factors (r) in such a way that the iteration process is stable and rapidly converging. Iteration procedures applying such a factor for better convergence are well known. They choose in most cases the "relaxation-factor" by an optimization problem looking for the minimum spectral radius of the iteration matrix. Applied to our problem we have to reduce the numerical set of equations (5) to a fixed-point form which represents a nonlinear point mapping set of equations [2].

To indicate the procedure and to achieve a better overview we modify the equations (5) a bit by putting $W_N \Rightarrow (W_N + W_R)$ and by assigning $u \Rightarrow u^{l+1} \Rightarrow (u^l + \Delta u)$. Inserting the equations of motion (first equation of (5)) into the contact velocity equations (second and third equations) and these into the constraints (the last two equations of (5)) we finally receive

$$\Lambda_N = \mathrm{prox}_{C_N}[(I - rG_{NN})\Lambda_N - rG_{NT}\Lambda_T - rw_N] \qquad \Lambda_T = \mathrm{prox}_{C_T}[(I - rG_{TT})\Lambda_T - rG_{TN}\Lambda_N - rw_T] \tag{6}$$

with the mass projection matrix $[G_{ij} = W_i^T M^{-1} W_j, \quad \text{(i,j)} = \text{(N,T)}]$ and the abbreviations $[w_i = W_i^T M^{-1}(u + h\Delta\text{t}), \quad \text{(i)} = \text{(N,T)}]$. The above equation can be interpreted as a fixed point equation of the form

$$\Lambda = \begin{pmatrix} \Lambda_N \\ \Lambda_T \end{pmatrix} = F(\Lambda) = \begin{pmatrix} \mathrm{prox}_{C_N}[(I - rG_{NN})\Lambda_N - rG_{NT}\Lambda_T - rw_N] \\ \mathrm{prox}_{C_T}[(I - rG_{TT})\Lambda_T - rG_{TN}\Lambda_N - rw_T] \end{pmatrix} \tag{7}$$

allowing the following fixed point iteration

$$\Lambda^{l+1} = F(\Lambda^l). \tag{8}$$

This iteration converges locally, if the spectral radius of $(\frac{\partial F}{\partial \Lambda})$ remains limited and smaller one, which can be realized by a corresponding optimization with respect to the factor (r) (for details see [2]).

4 Example roller coaster

One of the challenging practical examples concerning unilateral multibody theory are roller coasters. The trajectory configuration for such a roller coaster is depicted in Fig. 1 together with the vehicle system. The track generates frequent and abrupt changes of the vehicle's course resulting in a special thrill for the passengers. Thereby, and due to the slackness of the wheel rail contact, sequences of impacts with friction take place producing a significant amount of load and stress to the wheels and the undercarriage system. Each carriage possesses four packages each with six wheels coming out with altogether 24 wheel-track-contacts of a spatial character. Some experimental and simulation results will be presented.

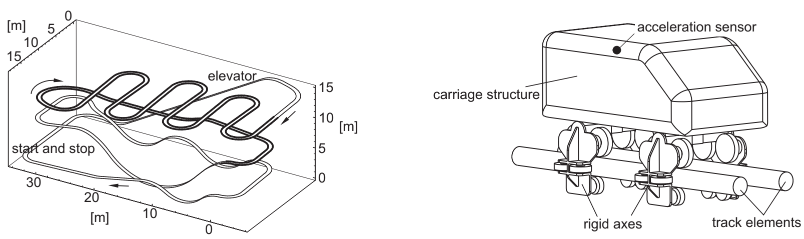

Figure 1: "Wild Mouse" and roller coaster carriage

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