CIRCULAR EDGE SINGULARITIES AND EDGE STRESS INTENSITY FUNCTIONS FOR THE ELASTICITY SYSTEM IN 3-D DOMAINS

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Summary Asymptotics of solutions to the elasticity system in the vicinity of a circular singular edge in a three-dimensional domain are derived and provided in an explicit form. These asymptotic solutions are represented by a family of eigen-functions with their shadows, and the associated edge stress intensity functions (EFIFs), which are functions along the circular edge. We provide explicit formulas for a penny-shaped crack for an axisymmetric case as well as a case in which the loading is non-axisymmetric. As a particular case we present explicitly the series expansion for a traction free or clamped penny-shaped crack in an axisymmetric or a non-axisymmetric situation. The precise representation of the asymptotic series allows to develop a new extraction technique based on the quasi-dual function method, for the edge stress intensity functions which are of practical engineering importance in predicting crack propagation..

INTRODUCTION AND NOTATIONS

The elastic solution in 2-D domains in non-smooth domains like polygons, is known to be described in terms of special singular functions depending on the geometry and the differential operators on one hand, and of unknown coefficients depending on the given right hand side and boundary conditions on the other hand. In three dimensional domains as polyhedral, for the straight edges we have provided explicit representation of the solutions [1] as a series characterized by: eigen-values α_i, their corresponding eigen-functions and shadow-functions $\boldsymbol{\Phi}_{il}(\varphi)$, and their corresponding edge stress intensity function $A_i(z)$, which are functions along the edge. Here we concentrate on circular edges (a "penny-shaped crack" being a special renown case) in 3-D domain (see a cross section in Figure 1), and derive explicitly singular series expansion in the vicinity of such an edge for the elasticity system. We provide explicitly the asymptotic solution for the elasticity system in the simplified case of the penny-shaped crack under axi-symmetric and non-axisymmetric boundary conditions. From the engineering perspective the edge stress intensity functions (ESIFs) $A_i(z)$, when $\alpha_i < 1$ are of major importance because these are correlated to failure initiation. For example $\alpha_1 = ½$ for the penny-shaped crack. This work is motivated by the need to compute ESIFs for elasticity problems in 3-D domains with curved cracks or re-entrant edges as shown in Figure 2. These are of significant engineering importance because the the ESIFs may (and often do) vary along the crack front. This presentation will provide the explicit representation of the asymptotic solution in the vicinity of circular edges in axi-symmetric and non-axisymmetric domains [2], and derive methods for the extraction of the edge stress intensity functions.

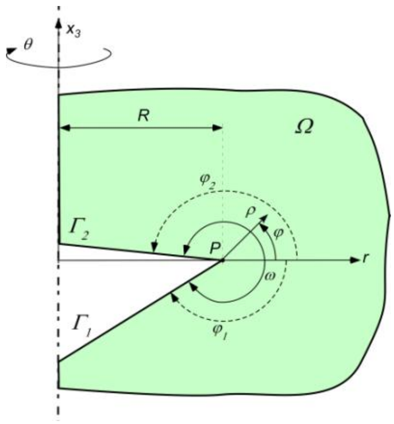

Fig. 1 *Model domain of interest Ω and the coordinate systems.*

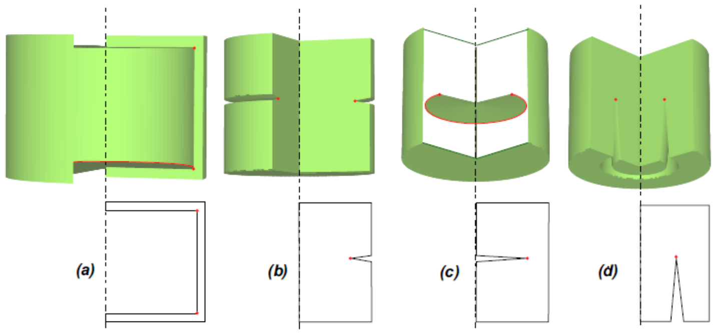

Fig. 2 *Different types of singular circular edges (only a sector is plotted so the circular edge is clearly visible. The domain in (c) includes the renown "penny-shaped" crack.*

THE NAVIER-LAME SYSTEM IN THE VICINITY OF A CURVED EDGE

Consider the $\rho,\varphi,\theta,$ coordinate system in the vicinity of the edge tip as shown in Figure 1. Expressing the Navier-Lame equations in these coordinates, and denoting the displacement vector by $(u_\rho,\, u_\varphi,\, u_\theta)^T$ one may show that these equations may be expressed as:

$$\Bigg[\left(1+\frac{\rho}{R}\cos\varphi\right)^2[M_0]+\left(1+\frac{\rho}{R}\cos\varphi\right)\left(\frac{\rho}{R}\right)[M_{01}]+\left(\frac{\rho}{R}\right)^2[M_{02}]$$
$$+\left(1+\frac{\rho}{R}\cos\varphi\right)\left(\frac{\rho}{R}\right)[M_{10}]\partial_\theta+\left(\frac{\rho}{R}\right)^2[M_{11}]\partial_\theta+\left(\frac{\rho}{R}\right)^2[M_2]\partial_{\theta\theta}\Bigg]u=\boldsymbol{0}$$

where the matrices $[M_0]$, $[M_{01}]$,... are given in terms of the material properties and $\rho,\varphi,$and their derivatives. This representation allows one to seek a solution as an asymptotic series of the form:

$$u = \sum_{\ell=0}\sum_{k=0} \partial_\theta^\ell A_k(\theta)\, \rho^{\alpha_k} \sum_{i=0}^{\infty} \left(\frac{\rho}{R}\right)^{i+\ell} \begin{Bmatrix} \phi^\rho(\varphi) \\ \phi^\varphi(\varphi) \\ \phi^\theta(\varphi) \end{Bmatrix}_{\ell,k,i} = \sum_{\ell=0}\sum_{k=0} \partial_\theta^\ell A_k(\theta)\, \rho^{\alpha_k} \sum_{i=0}^{\infty} \left(\frac{\rho}{R}\right)^{i+\ell} \phi_{\ell,k,i}$$

Substituting the asymptotic solution into the Navier-Lame system allows one to prove that the eigen-values and eigen-functions with *l=i=0* are the well known 2-D eigen-pairs, and the others (denoted shadow-functions) are computed by a recursive formulation. Finally, one may derive the full solution (displacements and stresses). As an example, for a penny-shaped crack under mode I loading in a non-axisymmetric case the stresses are (λ and μ are the Lame coefficients of the elastic material):

$$\begin{Bmatrix} \sigma_{\rho\rho} \\ \sigma_{\theta\theta} \\ \sigma_{\varphi\varphi} \\ \sigma_{\rho\theta} \\ \sigma_{\rho\varphi} \\ \sigma_{\theta\varphi} \end{Bmatrix} = \frac{K_I(\theta)}{\sqrt{2\pi\rho}} \left[\begin{pmatrix} -5\cos\frac{\varphi}{2} + \cos\frac{3\varphi}{2} \\ -\frac{4\lambda}{\lambda+\mu}\cos\frac{\varphi}{2} \\ -3\cos\frac{\varphi}{2} - \cos\frac{3\varphi}{2} \\ 0 \\ -\sin\frac{\varphi}{2} - \sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right) \begin{pmatrix} -\frac{5\lambda+13\mu}{4(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{\lambda+9\mu}{4(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ -\frac{2(2\lambda+\mu)(\lambda+5\mu)}{(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{3\lambda+2\mu}{\lambda+\mu}\cos\frac{3\varphi}{2} \\ -\frac{3(\lambda+9\mu)}{4(\lambda+\mu)}\cos\frac{\varphi}{2} - \frac{\lambda+9\mu}{4(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \\ \frac{\lambda-7\mu}{4(\lambda+\mu)}\sin\frac{\varphi}{2} + \frac{\lambda-7\mu}{4(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} \right.$$

$$\left. + \left(\frac{\rho}{R}\right)^2 \begin{pmatrix} -\frac{-33\lambda^2+430\lambda\mu+335\mu^2}{96(\lambda+\mu)^2}\cos\frac{\varphi}{2} - \frac{21\lambda^2+154\lambda\mu+53\mu^2}{36(\lambda+\mu)^2}\cos\frac{3\varphi}{2} + \frac{311\lambda-5\mu}{32(\lambda+\mu)}\cos\frac{5\varphi}{2} \\ \frac{\mu(23\lambda+19\mu)}{2(\lambda+\mu)^2}\cos\frac{\varphi}{2} - \frac{2(4\lambda+3\mu)(-3\lambda^2-10\lambda\mu+\mu^2)}{9(\lambda+\mu)^3}\cos\frac{3\varphi}{2} - \frac{3(5\lambda+4\mu)}{8(\lambda+\mu)}\cos\frac{5\varphi}{2} \\ \frac{111\lambda^2+142\lambda\mu-97\mu^2}{96(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{-3\lambda^2+74\lambda\mu+61\mu^2}{36(\lambda+\mu)^2}\cos\frac{3\varphi}{2} - \frac{3(3\lambda-13\mu)}{32(\lambda+\mu)}\cos\frac{5\varphi}{2} \\ 0 \\ \frac{105\lambda^2+322\lambda\mu+89\mu^2}{96(\lambda+\mu)^2}\sin\frac{\varphi}{2} + \frac{3\lambda^2+38\lambda\mu+19\mu^2}{12(\lambda+\mu)^2}\sin\frac{3\varphi}{2} + \frac{3(-9\lambda+7\mu)}{32(\lambda+\mu)}\sin\frac{5\varphi}{2} \\ 0 \end{pmatrix} + \cdots \right]$$

$$+ \frac{K_I'(\theta)}{\sqrt{2\pi\rho}}\left(\frac{\rho}{R}\right) \left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{2(\lambda-\mu)}{\lambda+\mu}\cos\frac{\varphi}{2} - \frac{2(\lambda+3\mu)}{\lambda+\mu}\cos\frac{3\varphi}{2} \\ 0 \\ \frac{2(\lambda+3\mu)}{\lambda+\mu}\sin\frac{\varphi}{2} + \frac{2(\lambda+3\mu)}{\lambda+\mu}\sin\frac{3\varphi}{2} \end{pmatrix} + \left(\frac{\rho}{R}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{9\lambda^2+96\lambda\mu+71\mu^2}{6(\lambda+\mu)^2}\cos\frac{\varphi}{2} - \frac{3(3\lambda+\mu)}{2(\lambda+\mu)}\cos\frac{3\varphi}{2} + \frac{9\lambda^2+24\lambda\mu+7\mu^2}{3(\lambda+\mu)^2}\cos\frac{5\varphi}{2} \\ 0 \\ \frac{(9\lambda+\mu)(3\lambda+5\mu)}{6(\lambda+\mu)^2}\sin\frac{\varphi}{2} + \frac{3(\lambda-\mu)}{2(\lambda+\mu)}\sin\frac{3\varphi}{2} - \frac{9\lambda^2+24\lambda\mu+7\mu^2}{3(\lambda+\mu)^2}\sin\frac{5\varphi}{2} \end{pmatrix} + \cdots \right.$$

$$+ \frac{K_I''(\theta)}{\sqrt{2\pi\rho}}\left(\frac{\rho}{R}\right)^2 \begin{pmatrix} \frac{-3\lambda+5\mu}{6(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{21\lambda+61\mu}{18(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ \frac{2(3\lambda+2\mu)}{\lambda+\mu}\cos\frac{\varphi}{2} - \frac{4(4\lambda+3\mu)(3\lambda+7\mu)}{9(\lambda+\mu)^2}\cos\frac{3\varphi}{2} \\ \frac{3\lambda-5\mu}{6(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{3\lambda-5\mu}{18(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \\ -\frac{3\lambda+11\mu}{6(\lambda+\mu)}\sin\frac{\varphi}{2} - \frac{3\lambda+11\mu}{6(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots$$

Having the eigen-pairs and their shadows available, we extend the quasi-dual function method presented in [1] for straight-edges problems to circular singular edges using the series expansion presented herein, and provide results extracted from finite element solutions.

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References

[1] Yosibash, Z., Omer, N., Costabel, M., Dauge, M.: Edge stress intensity functions in polyhedral domains and their extraction by a quasidual function method. *Int. Jour. Fracture*, **136**:37 – 73, 2005

[2] Yosibash Z., Shannon S., Dauge M. and Costabel M.: Circular edge singularities for the Laplace equation and the elasticity system in 3-D domains, *Int. Jour. Fracture*, **168**: 31–-52, 2011