THE EFFECT OF SURFACE ELASTICITY ON MORPHOLOGICAL INSTABILITY OF A STRESSED SOLID

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Summary To predict the relief formation in stressed solid, we use a linear stability analysis that takes into account surface elasticity. The total free energy of an elastic isotropic body is assumed to be the sum of the elastic strain energy and the surface energy that depends on surface deformation. These two quantities determine the chemical potential for mass transport via surface diffusion. A governing equation is derived that gives a change of the perturbation amplitude as a function of time.

The first theoretical investigation of the morphological instability in stressed solids dates back to the work of Asaro and Tiller [1]. They found that the flat surface is unstable under perturbations in shape if wavelength is greater than the some critical value. The critical wavelength is proportional to the ratio of the constant surface energy to the local elastic energy. The rate at which these unstable shape perturbations grow is controlled by surface diffusion. The same result was later independently received by Grinfeld [2] and Srolovitz [3]. Grilhe [4] considered constant elastic stress (surface tension) and found that conditions for stability of the surface are sensitive to the sign of applied stress. Later, Wu et al. [5], using an experimentally justified universal binding-energy-distance curve, consider surface deformation depending on surface stress and surface energy. The stability condition of the surfaces of stressed solids was re-examined using the revised equations. However, the surface elastic constants were neglected.
In this paper we try to investigate the effect of surface stress on morphological instability of a stressed surface based on surface continuum mechanics of Gurtin and Murdoch [6]. Consider isotropic stressed solid under plane strain conditions. The perturbation of the free surface is assumed to take the form

$$h(x_1,t) = A(t)f(x_1), \tag{1}$$

where $|f(x_1)| \le 1,\ \ f(x_1) = f(-x_1) = f(x_1+\lambda),\ A(t)/\lambda = \varepsilon \ll 1,\ A(0) = A_0 \ne 0$ is perturbation amplitude.
The bulk material is idealized as an elastic semi-infinite continuum and the surface as a pre-stressed elastic membrane coherently bonded to the boundary of the bulk material

$$\lim_{z\to z_s} u(z) = u^s(z_s). \tag{2}$$

Here $u = u_1 + iu_2$ is the displacement vector, i is the imaginary unit.
Therefore, constitutive equations for bulk and surface phase are consequently

$$\sigma_{nn} = (\lambda+2\mu)\varepsilon_{nn} + \lambda\varepsilon_{tt},\ \ \sigma_{tt} = (\lambda+2\mu)\varepsilon_{tt} + \lambda\varepsilon_{nn},\ \ \sigma_{nt} = 2\mu\varepsilon_{nt},\ \sigma_{33} = \lambda(\varepsilon_{nn}+\varepsilon_{tt}), \tag{3}$$

$$\sigma^s_{tt} = \sigma^{s0}_{tt} + (\lambda_s + 2\mu_s)\varepsilon^s_{tt},\ \sigma^s_{33} = \sigma^{s0}_{33} + \lambda_s\varepsilon^s_{tt}. \tag{4}$$

In eqs. (4), $\lambda_s,\ \mu_s$ are the surface constants similar to the bulk Lame constants λ, μ in eqs. (3); $\sigma^{s0}_{tt},\ \sigma^{s0}_{33}$ are the residual surface stresses.
The equilibrium condition of curved surface is described by the generalized Young-Laplace equation which in two-dimensional case can be written as [7]

$$\sigma_n(z_s) = \sigma_{nn}(z_s) + i\sigma_{nt}(z_s) = \kappa\sigma^s_{tt}(z_s) - i\frac{d\sigma^s_{tt}(z_s)}{ds},\ z_s \in S, \tag{5}$$

where κ is a local principal curvature. The boundary conditions at infinity are

$$\lim_{x_2\to-\infty}(\sigma_{22} - i\sigma_{12}) = 0,\ \ \lim_{x_2\to-\infty}\omega = 0,\ \lim_{x_2\to-\infty}\sigma_{11} = \sigma^{\infty}_{11}, \tag{6}$$

where ω is a rotation angle of a material particle.
Total free energy of the solid at constant temperature is taken as the sum of the elastic strain energy and surface energy

$$E(t) = \int_B U(\varepsilon_{ij})dB + \int_S U_s(\varepsilon^s_{tt})dS, \tag{7}$$

where B is the volume of the bulk phase and S is the boundary.
The change of free energy due to change of the surface shape described by local normal velocity v_n is

$$\frac{\partial E(t)}{\partial t} = \int_B \frac{\partial U}{\partial t}dB + \int_S Uv_n dS + \int_S \frac{\partial U_s}{\partial t}dS - \int_S \kappa U_s v_n dS. \tag{8}$$

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Since we are dealing with a conservative mechanical system, the change of free energy reduces to

$$\frac{\partial E(t)}{\partial t} = \int_S (U - \kappa U_s)\, v_n dS. \tag{9}$$

Thus, the local chemical potential for the surface can be defined as

$$\mu(z_s, t) = (U(z_s, t) - \kappa(x_1, t) U_s(z_s, t))\, \Omega, \tag{10}$$

where Ω is atomic volume.
The variation in chemical potential along the curved surface gives a rise of atoms along the surface

$$J = -\frac{D_s C_s}{k_b T} \frac{\partial \mu}{\partial s}, \tag{11}$$

where D_s is the surface self-diffusivity, C_s is the number of diffusion atoms per unit area, k_b is the Boltzmann constant, and T is absolute temperature.
Conservation of mass at each point along the surface gives the normal velocity

$$v_n = -\Omega \frac{\partial J}{\partial s}. \tag{12}$$

Under expressions (1), (10) and (11), the surface evolution equation (12) takes the form

$$\frac{\partial h(x_1, t)}{\partial t} = \frac{K}{1 + h'^2(x_1, t)} \frac{\partial^2}{\partial x_1^2} \left[U(z_s, t) - \frac{h''(x_1, t)}{(1 + h'^2(x_1, t))^{3/2}} U_s(z_s, t) \right], \; K = \frac{D_s C_s \Omega}{k_b T}. \tag{13}$$

In order to integrate this equation, we solve the corresponding boundary value problem (2), (5) and (6) for finding surface energy density U_s and elastic strain energy density U at the curved surface (1). Using the perturbation technique, a corresponding variation of a stress field is obtained in first-order approximation

$$\sigma_{tt}^s = \sigma_{tt}^{s(0)} + \varepsilon \sigma_{tt}^{s(1)} + O(\varepsilon^2), \; \varepsilon_{tt} = \varepsilon_{tt}^{s(0)} + \varepsilon \varepsilon_{tt}^{s(1)} + O(\varepsilon^2), \tag{14}$$

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \varepsilon \sigma_{ij}^{(1)} + O(\varepsilon^2), \; \varepsilon_{ij} = \varepsilon_{ij}^{(0)} + \varepsilon \varepsilon_{ij}^{(1)} + O(\varepsilon^2). \tag{15}$$

Taking into account the continuity of the displacement (2) and Hooke's law (4), we can write the boundary equation

$$\sigma_{tt}^s = \sigma_{tt}^{s0} + (\lambda_s + 2\mu_s)\, \varepsilon_{tt}(z_s). \tag{16}$$

Based on Goursat-Kolosov's complex potentials, Musskhelishvilli's representation, Taylor seeries and properties of Cauchy-type integral, we obtain a sequence of hypersingular integral equations in the unknown expansion coefficients $\sigma_{tt}^{s(k)}$

$$\sigma_{tt}^{\prime s(k)}(x_1) - \frac{M(\eta + 1)}{2\pi} \int_{-\infty}^{+\infty} \frac{\sigma_{tt}^{\prime s(k)}(\xi)}{(\xi - x_1)^2} d\xi = H_k(x_1), \; k = \overline{0, 1}. \tag{17}$$

In (17), there are introduced the following notations $M = \dfrac{(\lambda_s + 2\mu_s)}{2\mu}$ and $\eta = 3 - 4\nu$. Here $H_0 = 0$ and H_1 is known function which depends on physical and geometrical parameters, ν is Poisson's ratio of a bulk material. A solution of this problem is obtained by expansion in a Fourier series.
After that, an arbitrary surface profile described by a periodic function (1) is also represented by a Fourier series. A derived governing equation (13) gives the amplitude of a surface roughness as a function of time. And the stability condition can be formulated like this: if the amplitude of waviness decreases with time, the flat surface of the film is stable. A parametric study is then carried out to investigate the influence of various practically important parameters on the critical wavelength.

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References

[1] Asaro R.J., Tiller W.A.: Interface morphology development during stress corrosion cracking: part 1. Via surface diffusion. *Metal. Trans.* **3**:1789–1796, 1972.
[2] Grinfeld M.A.: Instability of the separation boundary between a non-hydrostatically stressed elastic body and melt. *Sov. Phys. Dokl.* **31**: 831–834, 1986.
[3] Srolovitz D.J.: On the stability of surfaces of stressed solids. *Acta Metall.* **37**: 621–625, 1989.
[4] Grilhe J.: Study of roughness formation induced by homogeneous stress at the free surface of solids. *Acta Metall.* **41**: 909–913, 1993.
[5] Wu C.H., Hsu J., Chen C.-H.: The effect of surface stress on the stability of surfaces of stressed solids. *Acta Metall.* **46**: 3755–3760, 1998.
[6] Gurtin M.E., Murdoch A.I.: A continuum theory of elastic material surfaces. *Arch. Ration. Mech. Anal.* **57**: 291–323, 1975.
[7] Grekov M.A., Kostyrko S.A. Instability of a flat surface of a film coating due to surface diffusion. *Vestnik St.-Persburg Univ.* Ser. 10, No. 1: 47–54, 2007. (in Russian).