STRAIN GRADIENT SOLUTIONS FOR HALF-SPACE CONTACT PROBLEMS

X.-L. Gao[* a)] and S. S. Zhou[**]
[*]*Department of Mechanical Engineering, University of Texas at Dallas,*
800 West Campbell Road, Richardson, TX 75080-3021, USA
[**]*Department of Mechanical Engineering, Texas A&M University,*
College Station, TX 77843-3123, USA

Summary A general solution for the stress and displacement fields in an elastic half-space loaded by a concentrated or distributed normal force is derived using a simplified strain gradient elasticity theory. Several representative contact problems, including Flamant and Boussinesq problems, are studied by applying the general solution derived. Axisymmetric indentation problems involving different indenter profiles (flat-ended, spherical and conical) are also analyzed using the current solutions.

INTRODUCTION

Due to a lack of material length scale parameters, classical continuum theories cannot explain size effects exhibited by many materials at micron and nano-meter scales. The simplified strain gradient elasticity theory (SSGET) [1,2], also known as the first gradient elasticity theory of Helmholtz type and the dipolar gradient elasticity theory, contains one material length scale parameter and can capture such microstructure-dependent size effects. The SSGET is the simplest strain gradient elasticity theory evolving from Mindlin's pioneering work [3,4] and has been successfully used to solve a number of important problems on dislocations, cracking, wave dispersion, inclusions, beams, plates, and thick-walled shells.

In the present work, a general solution for the stress and displacement fields in an elastic half-space loaded by a concentrated or distributed normal force is derived using the SSGET. The displacement function method of Mindlin [3] is employed, and the solutions of the half-space and half-plane boundary-value problems are analytically obtained in integral forms using the Fourier and Hankel transform techniques. Several representative contact problems, including Flamant and Boussinesq problems, are studied by applying the solutions derived. Axisymmetric indentation problems involving different indenter profiles (flat-ended, spherical and conical) [5] are also analyzed using the current solutions. It is shown that the newly obtained solutions can capture the size dependence of hardness, which is known as the indentation size effect [6,7]. In addition, it is found that the elastic fields given by the current strain gradient solutions change smoothly and do not exhibit the singularities and discontinuities predicted by the counterpart solutions based on classical elasticity.

RESULTS AND DISCUSSION

Indentation hardness can be defined as the ratio of applied load P to indentation depth (or penetration of the indenter tip) δ:

$$H = \frac{P}{\delta}, \tag{1}$$

which is known as the Rockwell hardness.

From Eq. (1) and the expressions of P and δ based on classical elasticity, the indentation hardness has been determined to be

$$H_B^c = \frac{4\mu a}{1-\nu}, \quad H_H^c = \frac{8\mu a}{3(1-\nu)}, \quad H_C^c = \frac{2\mu a}{1-\nu} \tag{2}$$

for a Boussinesq flat-ended punch, a Hertzian spherical punch and a conical punch, respectively.

By using the P and δ expressions derived in the current solution based on the SSGET, Eq. (1) gives

$$\frac{H_B^c}{H_B} = 1 - \frac{2}{\pi}\int_0^\infty \psi(a,t)t^{-1}\sin t dt \tag{3a}$$

for the Boussinesq flat-ended punch,

$$\frac{H_H^c}{H_H} = 1 - \frac{4}{\pi}\int_0^\infty \psi(a,t)t^{-3}\left(\sin t - t\cos t\right) dt \tag{3b}$$

for the Hertzian spherical punch, and

a) Corresponding author. Email: Xin-Lin.Gao@utdallas.edu.

$$\frac{H_C^c}{H_C} = 1 - \frac{2}{\pi}\int_0^\infty \psi(a,t) t^{-2}\left(1-\cos t\right) dt \tag{3c}$$

for the conical punch, where H_B^c, H_H^c and H_C^c are given in Eq. (2), and

$$\psi(a,t) = \frac{\left[2(1-\nu) + t\tilde{\zeta}^{-1} - 2\tilde{l}^4 t\tilde{\zeta}(\tilde{\zeta}-t)^2\right]\tilde{l}^2 t^2}{(1-\nu)(1+2\tilde{l}^2 t^2) + 2\tilde{l}^6\tilde{\zeta}^2 t^2(\tilde{\zeta}-t)^2},\ \tilde{l} = L/a,\ \tilde{\zeta} = \sqrt{1+\tilde{l}^2 t^2}, \tag{3d}$$

with L being the material length scale parameter included in the SSGET and a being the radius of the projected contact area (i.e., indentation radius).

Clearly, Eqs. (3a–d) show that the indentation hardness in each case depends on the material microstructure as measured by the length scale parameter L and the indentation radius a, thereby capturing the indentation size effect. When the microstructural effect is not considered, L = 0 and H_B, H_H and H_C based on the SSGET reduce to their classical elasticity-based counterparts H_B^c, H_H^c and H_C^c. Also, for macro-indentation tests with $L / a << 1$, $\psi(a, t) \to 0$ and thus the indentation hardness value predicted by the current solution approaches from above to that given by the classical elasticity-based solution. However, the indentation size effect is significant for micro-indentations tests with small a/L values, as illustrated in Figure 1.

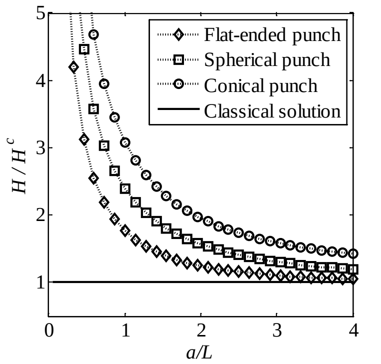

Figure 1. Indentation hardness curves for the three punch shapes.

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