TIME-AVERAGED ACOUSTIC FORCES ACTING ON A CIRCULAR CYLINDER IN A VISCOUS FLUID

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Summary The time-averaged acoustic force acting on a cylinder in a viscous fluid is formulated analytically by the perturbation method with assumptions such as low incident amplitude and thin acoustical boundary layer. The inertia effects are taken into account in the second-order calculation to avoid the difficulty of Stokes' paradox. The results compare well with the former theories.

INTRODUCTION

Recently, the time-averaged force acting on a particle submerged in a sound field, which derives from the nonlinear interaction with the host fluid, becomes more and more attractive due to its applications in ultrasonic particle manipulators [1]. Undoubtedly, it is very crucial to accurately predict the actual forces and torques acting on the particles generated by the incident ultrasonic waves to design and optimize ultrasonic particle manipulators. Many pioneers, e.g. King [2], Gor'kov [3], Doinikov [4], Wang [5] and Bruus [6], have addressed this topic under specific conditions. Most of the former investigations were focused on axisymmetric acoustic forces acting on a sphere or a cylinder in an inviscid (King and Gor'kov) or a viscous (Doinikov, Wang and Bruus) fluid. Although Doinikov has proposed a general theory to consider fluid viscosity, the convential Stokes' paradox arises when his theory is applied to two-dimensional calculations. In this paper, an analytic general formulation for the time-averaged force for two-dimensional problems is established based on the perturbation method under the conditions of small incident amplitude and thin acoustical boundary layer. The solution is more general and accurate than Wang [5]. Moreover, it can also be extended to the three-dimensional situation and is certainly more accurate than Doinikov's theory [4]. The formulation presented here provides more insightful physical understanding to the time-averaged acoustic force and may become a theoretical basis for numerical modeling to compute the acoustic force in practical cases such as irregular particle shapes and complex device boundaries.

THEORY

Governing equations

The dimensionless governing equations for a viscous flow motion without heat conduction are

$$\left\{\begin{array}{c} M_p^2\dfrac{\partial\rho}{\partial t}+\nabla\cdot(1+\varepsilon M_p^2\rho)\mathbf{v}=0 \\ (1+\varepsilon M_p^2\rho)\left(\dfrac{\partial\mathbf{v}}{\partial t}+\varepsilon\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla p+\dfrac{1}{M^2}\left[\nabla^2\mathbf{v}+\dfrac{1}{3}\nabla\nabla\cdot\mathbf{v}\right]\end{array}\right. \tag{1}$$

with the linear equation of state $p=\rho$, where the dimensionless parameter $M_p^2=\omega^2a^2/c_0^2$ indicating the compressibility, $\varepsilon=U_\infty/(\omega a)$ showing the convection effects, $M^2=\omega a^2/\nu$ denoting the viscous effects and U_∞ is the velocity amplitude of the incident wave. Here, the normal cases with $\varepsilon\ll 1$ and $M^2\gg 1$ are considered. By expanding the variables as well as the governing equations with the small parameter ε, we have the equations for the variables of order ε^0 and for the time-averaged variables of order ε^1, respectively

$$\left\{\begin{array}{c} M_p^2\dfrac{\partial\rho_1}{\partial t}+\nabla\cdot\mathbf{v}_1=0 \\ \dfrac{\partial\mathbf{v}_1}{\partial t}=-\nabla p_1+\dfrac{1}{M^2}\left[\nabla^2\mathbf{v}_1+\dfrac{1}{3}\nabla\nabla\cdot\mathbf{v}_1\right]\end{array}\right. \quad\text{and}\quad \left\{\begin{array}{c}\nabla\cdot\mathbf{v}_2=0 \\ \varepsilon^2\mathbf{v}_2\cdot\nabla\mathbf{v}_2=-\nabla p_2+\dfrac{1}{M^2}\nabla^2\mathbf{v}_2-\mathbf{F}_R\end{array}\right., \tag{2}$$

where the first-order equation of state $p_1=\rho_1$ and $\mathbf{F}_R=\langle\mathbf{v}_1\nabla\cdot\mathbf{v}_1+\mathbf{v}_1\cdot\nabla\mathbf{v}_1\rangle$ is the Reynolds stress. The inertia term in the right equation in Eq. (2) can be neglected only in the near field and must be take into account in far field for two-dimensional flow motions. The time-averaged force acting on the cylinder per length is

$$\mathbf{F}=\int_{S_0}\langle\boldsymbol{\sigma}_2-\mathbf{v}_1\otimes\mathbf{v}_1\rangle\cdot\mathbf{n}\mathrm{d}S, \tag{3}$$

where $\boldsymbol{\sigma}_2=-\varepsilon p_2\mathbf{I}+(\varepsilon/M^2)[\nabla\mathbf{v}_2+(\nabla\mathbf{v}_2)^{\mathrm{T}}]$ is the time-averaged stress tensor of order ε^1.

First-order field

The first-order governing equations are equivalent to the following two Helmholtz equations for velocity potentials

$$(\nabla^2+k_c^2)\varphi=0 \text{ and } (\nabla^2+k_s^2)\psi=0, \tag{4}$$

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with the boundary condition $\mathbf{v}_1 = (\mathrm{i}/m_p)\int_{S_0} \boldsymbol{\sigma}_1 \cdot \mathbf{n}\mathrm{d}S$ at $r=1$, where $\boldsymbol{\sigma}_1 = [-p_1 - 2/(3M^2)\nabla\cdot\mathbf{v}_1]\mathbf{I} + (1/M^2)[\nabla\mathbf{v}_1 + (\nabla\mathbf{v}_1^{\mathrm{T}})]$ and m_p is the particle mass. The total velocity potentials are expressed in cylindrical coordinates by

$$\varphi = \sum_{n=-\infty}^{\infty} C_n[J_n(k_c r) + \alpha_n H_n^{(1)}(k_c r)]\mathrm{e}^{\mathrm{i}n\theta} \quad \text{and} \quad \psi = \sum_{n=-\infty}^{\infty} C_n\beta_n H_n^{(1)}(k_s r)\mathrm{e}^{\mathrm{i}n\theta}, \tag{5}$$

where C_n are the expanding coefficients of the incident wave and α_n and β_n are the scattering coefficients determined by the boundary conditions of Eq. (4).

Second-order field

By introducing the stream function $\mathbf{v}_2 = \nabla \times \Psi\mathbf{e}_z$ and the Helmholtz decomposition for the Reynolds stress $M^2\mathbf{F}_R = \nabla q + \nabla \times Q\mathbf{e}_z$, we have

$$\nabla^4\Psi - \beta^2\nabla^2\Psi = \nabla^2 Q. \tag{6}$$

with the boundary conditions $\mathbf{v}_2 \to \mathbf{0}$ as $r \to \infty$ and $\mathbf{v}_2 = -\mathbf{U}_2 = -\langle\left(\int \mathbf{v}_1\mathrm{d}t\right)\cdot\nabla\mathbf{v}_1\rangle$ at $r=1$. Here, the inertial term has been linearized by $\mathbf{v}_2\cdot\nabla\mathbf{v}_2 \approx \mathbf{U}_{2I}\cdot\nabla\mathbf{v}_2$ with $\mathbf{U}_{2I} = \langle\left(\int \mathbf{v}_{1I}\mathrm{d}t\right)\cdot\nabla\mathbf{v}_{1I}\rangle$ and β is dependent on $\mathbf{U}_{2I}$, where $\mathbf{v}_{1I}$ denotes the incident wave velocity. By assuming

$$\Psi = \sum_{n=-\infty}^{\infty} \Psi_n(r)\mathrm{e}^{\mathrm{i}n\theta}, \quad q = \sum_{n=-\infty}^{\infty} q_n(r)\mathrm{e}^{\mathrm{i}n\theta} \quad \text{and} \quad Q = \sum_{n=-\infty}^{\infty} Q_n(r)\mathrm{e}^{\mathrm{i}n\theta}, \tag{7}$$

one can obtain the solution of Eq. (6), for example $n=1$, as

$$\begin{aligned}\Psi_1(r) = &-\frac{r}{2\beta^2}\left[\int_1^r x^{-2}(\mathrm{i}b_{1,1} - b_{2,1})\mathrm{d}x + C_{1,1}\right] + \frac{r^{-1}}{2\beta^2}\left[\int_1^r (\mathrm{i}b_{1,1} + b_{2,1})\mathrm{d}x + C_{2,1}\right] \\ &+ \frac{1}{2\beta}I_1(\beta r)\left[\int_1^r K_2(\beta x)(\mathrm{i}b_{1,1} - b_{2,1})\mathrm{d}x - \int_1^r K_0(\beta x)(\mathrm{i}b_{1,1} + b_{2,1})\mathrm{d}x + C_{5,1}\right] \\ &+ \frac{1}{2\beta}K_1(\beta r)\left[\int_1^r I_2(\beta x)(\mathrm{i}b_{1,1} - b_{2,1})\mathrm{d}x - \int_1^r I_0(\beta x)(\mathrm{i}b_{1,1} + b_{2,1})\mathrm{d}x + C_{6,1}\right],\end{aligned} \tag{8}$$

where the parameter $b_{1,n}$ and $b_{2,n}$ are defined by the Reynolds stress $rM^2\mathbf{F}_R = \sum(\mathbf{e}_r b_{1,n} + \mathbf{e}_\theta b_{2,n})\mathrm{e}^{\mathrm{i}n\theta}$. The coefficients in Eq. (8) are determined by the boundary conditions as $C_{1,1} = -\int_1^\infty x^{-2}(\mathrm{i}b_{1,1} - b_{2,1})\mathrm{d}x$, $C_{5,1} = \int_1^\infty K_0(\beta x)(\mathrm{i}b_{1,1} + b_{2,1})\mathrm{d}x - \int_1^\infty K_2(\beta x)(\mathrm{i}b_{1,1} - b_{2,1})\mathrm{d}x$, $C_{2,1} = \{[\beta K_0(\beta) + 2K_1(\beta)]C_{1,1} - \beta C_{5,1}\}/[\beta K_0(\beta)] - 2\mathrm{i}\beta^2\{1 + K_1(\beta)/[\beta K_0(\beta)]\}U_{2r,1} - 2\beta[K_1(\beta)/K_0(\beta)]U_{2\theta,1}$ and $C_{6,1} = [\beta^2 I_0(\beta)C_{5,1} - 2C_{1,1}]/[\beta^2 K_0(\beta)] + [2\mathrm{i}/K_0(\beta)]U_{2r,1} + [2/K_0(\beta)]U_{2\theta,1}$. The solution for $n=-1$ can be obtained in the same manner.

Time-averaged force

The time-averaged forces acting on the cylinder per length in *x*- and *y* - directions are then calculated by

$$F_x = \pi\mathrm{Re}[\sigma_{2trr,-1} + \sigma_{2trr,1} + \mathrm{i}\sigma_{2tr\theta,-1} - \mathrm{i}\sigma_{2tr\theta,1}] \quad \text{and} \quad F_y = \pi\mathrm{Re}[-\mathrm{i}\sigma_{2trr,-1} + \mathrm{i}\sigma_{2trr,1} + \sigma_{2tr\theta,-1} + \sigma_{2tr\theta,1}] \tag{9}$$

where $\sigma_{2trr,n} = \sigma_{2rr,n} - \sigma_{1arr,n}$ and $\sigma_{2tr\theta,n} = \sigma_{2r\theta,n} - \sigma_{1ar\theta,n}$. The parameters $\sigma_{2rr,n}$ and $\sigma_{2r\theta,n}$ are related to the second-order velocity achieved in the last subsection and $\sigma_{1arr,n}$ and $\sigma_{1ar\theta,n}$ are determined by the time-averaged first-order momentum flux with $\langle\mathbf{v}_1\otimes\mathbf{v}_1\rangle_{rr} = \sum\sigma_{1arr,n}\mathrm{e}^{\mathrm{i}n\theta}$ and $\langle\mathbf{v}_1\otimes\mathbf{v}_1\rangle_{r\theta} = \sum\sigma_{1ar\theta,n}\mathrm{e}^{\mathrm{i}n\theta}$.

EXAMPLE

A rigid cylinder of radius $5\times10^{-5}\mathrm{m}$ is fixed in the host medium with the density $1000\mathrm{kg/m^3}$ and the speed of sound $1400\mathrm{m/s}$. The incident planar traveling wave is of density amplitude $0.1\mathrm{kg/m^3}$ and wave length $5\times10^{-4}\mathrm{m}$ corresponding to a frequency of $2.8\mathrm{MHz}$. The acoustic radiation force is given as $8.787\times10^{-4}\mathrm{N/m}$ in an ideal fluid by using the theory in Ref. [7]. The total time-averaged force predicted by the current theory including viscosity is $9.063\times10^{-4}\mathrm{N/m}$ which is close to the results in Ref. [5].

References

[1] Oberti S.: Micromanipulation of small particles within micromachined fluidic systems using ultrasound. *PhD Thesis*, ETH Zurich, No. 18452, 2009.

[2] King L.V.: On the acoustic radiation pressure on spheres. *Proc. R. Soc. Lond. A* **147**:212–240, 1934.

[3] Gor'kov L.P.: On the forces acting on a small particle in an acoustical field in an ideal fluid. *Sol. Phys.* **6**:773–775, 1962.

[4] Doinikov A.A.: Acoustic radiation force on a spherical particle in a viscous heat-conducting fluid. I. General formula. *J. Acoust. Soc. Amer.* **101**: 713-721, 1997.

[5] Wang J., Dual J.: Theoretical and numerical calculations for the time-averaged acoustic force and torque acting on a rigid cylinder of arbitrary size in a low viscosity fluid. *J. Acoust. Soc. Amer.* **129**: 3490–3501, 2011.

[6] Bruus H., Settnes H.: Acoustophoresis of suspended spherical microparticles: the acoustic radiation force with viscous corrections. *ICU 2011*.

[7] Wang J., Dual J.: Numerical simulations for the time-averaged acoustic forces acting on rigid cylinders in ideal and viscous fluids. *J. Phys. A* **42**: No. 285502, 2009.