THEORIES OF OPTIMAL ROTOR

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Summary The purpose of this work is to reexamine of the key rotor theories with ideal load distribution, to clarify some of the more obscure and fallacious points and treatments of its, to present a systematic optimizing and design procedure, and to provide for the first time an accurate comparison of the optimum rotor theories that are milestones in development of the rotor aerodynamics.

Introduction

The literature of rotor aerodynamics is scattered and in some respects is inconsistent and incomplete for example it concerns optimization of complex system such as rotor blades. The problem is in general quite simple to formulate: for example in wind energy the objective is to design the wind turbine which produces the highest possible power for given wind. There are many different theories of the optimal rotor but most optimization approaches have not had a clear systematic procedure to make a correct comparison between them. The purpose of this work is to reexamine of the key rotor theories to find a simple and universal algorithm for the rotor optimization and design.

Key rotor theories with an ideal load distribution

The development of a rational rotor theory begins in the 19th century with fundamental momentum relations by Rankine [1] and Froude [2] which well-known as theory of an actuator disk now. However the clearer understanding of a connection between an action of real rotor blades and the actuator disc abstraction has been afforded by vortex rotor theory created independently by Joukowsky and Betz in early of the 20th century only. Then an evident result of the maximum efficiency in 59.3% for wind turbines named now as *Betz-Joukowsky limit* [3] have been concluded from the actuator disc theory directly.

At the end of the 19th century Drzewiecki [4] presented rotor theory via the blade elements as individual lifting surfaces. He took no account of the effect on each element of the velocity induced by the rotor itself. The combined blade element and momentum theory by Glauert [5] succeeded in estimation of the optimum rotor and corrected the Betz-Joukowsky limit for different operating regimes of wind turbines. Though the *Glauert's rotor optimization* depended on a principle of ‘‘independence of blade elements’’ which eventually came to be understood without physical justification it was more often employed in practical calculations because during long time it seems to have been the single real practical design procedure to optimal rotor.

The vortex rotor theory created independently by Joukowsky [6] and Betz [7] is analogous to Prandtl's wing theory in that the rotor blade is considered to be a lifting surface about which there is a circulation associated with the bound vorticity and a vortex sheet is continuously shed from the trailing edge. Unfortunately for rotors with finite number of blades both authors have outlined only conceptual ideas of different vortex structures (fig. 1) in wake, whereas his theoretical result concerned actuator disk theory that may be interpreted as the vortex theory with infinite number of blades and with constant circulation (*Joukowsky rotor*) or with helical vortex sheet moving uniform (*Betz rotor*).

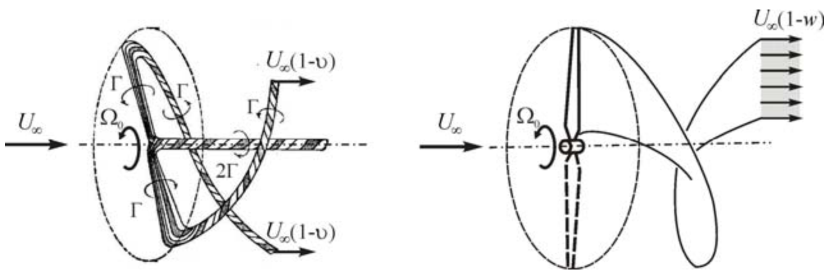

Fig 1 Sketch of the vortex systems corresponding to the lifting line theories proposed by Joukowsky (left) for rotor with a *constant* circulation along blade and Betz (right) for rotor with distribution of circulation along blades which generates rigidly helicoidal wake that moves in the axial direction with a *constant* velocity.

After several years later the model of the Joukowsky rotor has been proved by his pupils to be incorrect [8] and the last discussion about the troubles of his model has been argued in [9]. Betz determined the best distribution of trust (or circulation) along radial direction of actuator disk in his vortex model and Prandtl found an approximate solution (*Prandtl tip correction*) to expand the Betz solution to the case of finite number of blades by associative task for 2-D flow (see appendix to [6]). Goldstein [10] has given an accurate solution of the problem for the potential field and the distribution of circulation of the helicoidal vortex system coincided to case of the finite number of blades. He presented values of the circulation distribution only for 2- and 4-bladed propellers (see also [11]) which we named as *Goldstein's rotor solution* to distinct it from Betz rotor solution. The application of all above solutions was limited to “lightly loaded rotor” when an influence of the induction velocity by vorticity in the wake on them formation was neglected.

The next *Theodorsen's rotor solution* [12] includes the influence of the induction velocity on the wake formation in the Goldstein's solution. He and authors of [13] showed that the light loading limitation can be removed but both considerations have used the induction velocity of far wake and contained a complex problem for the rotor optimization with an uncertainty when the unknown wake structure should affect via an undefined induction velocity to create itself.

The recent development of the rotor models [14] (*OS rotors with constant* and *Goldstein's circulations*) for the first time provided consistent solutions for both vortex systems in wake behind rotor with finite number of blades (fig. 1).

Principal assumptions for the different rotor theories

Theories	Number of blades	Definition of the pitch in the wake	Circulation along blade
Betz-Joukowsky limit	actuator disk	non specified	non specified
Glauert's rotor optimization	non specified	non specified*)	non specified
Joukowsky rotor	infinite	without induction velocity	constant
Betz rotor	infinite	without induction velocity	Betz type
Prandtl tip correction	finite	without induction velocity	correction of Betz type
Goldstein's rotor solution	finite	without induction velocity	Goldstein's type
Theodorsen's rotor solution	finite	with induction velocity of far wake	Goldstein's type
OS rotor with constant circulation	finite	with induction velocity in rotor plane	constant
OS rotor with Goldstein's circulation	finite	with induction velocity in rotor plane	Goldstein's type

*) Instead of the wake specification the axial and azimuthal induction factors were introduced in rotor plane!

Rotor optimization

The practical procedure for design of the optimal rotor bases on a correct choice of a unique parameter for the rotor optimization with which a simple conversion from a selected wake pitch to the appropriate operating parameter (tip speed ratio) may be done. For two rotor types considered in [14] these parameters were found to be an average axial induction velocity $\bar{a}$ in rotor plane for Joukowsky vortex system (fig. 1, left) and induction part $\bar{w}$ of the total velocity of axial moving for the Betz wake (fig. 1, right). In present work this procedure was proved as a universal approach for all rotor theories under windmill operating regimes. The results of the optimizations present on fig. 2.

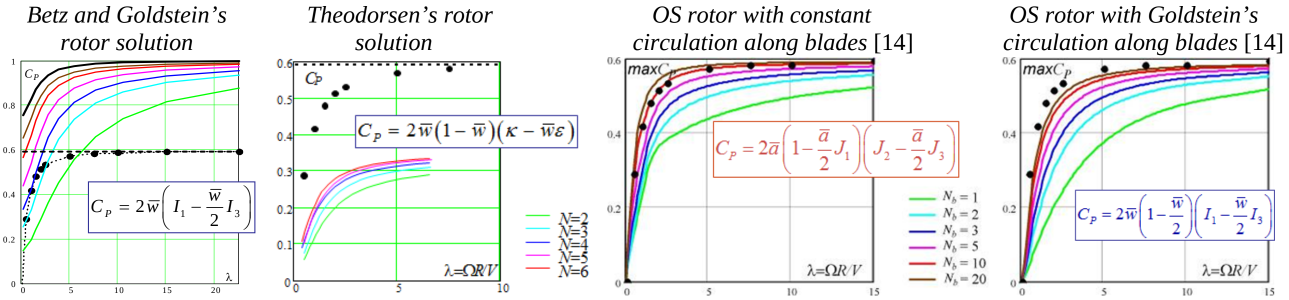

Fig 2 Comparison of the different rotor models under optimal windmill operations: points on all plots results *Glauert's rotor optimization*; collar curves indicates optimum power coefficient C_P for all models and black one on the first plot for *Betz rotor*.

From the comparison we conclude that all rotor models limited by "lightly loaded rotor" (first plot of fig. 2) are incorrect to predict maximum efficiency much more than *Betz-Joukowsky limit*. The *Theodorsen's rotor solution* gives values smaller in two times (second plot of fig. 2) and as it was mentioned above the model of the *Joukowsky rotor* is wrong too. It seems that only four rotor theories indicated in first two and last two lines of the table have good agreed with each other (third and fourth plots of fig. 2).

CONCLUSIONS

As result of the current study the fundamental results of the rotor aerodynamics (*actuator disk theory*, *Betz-Joukowsky limit Glauert's rotor optimization*) have been maintained and extended by new analytical solutions for optimal rotor with an ideal load distribution [14]. During long time when they were created there were many mistaken attempts to revise them (for example it concerns some false modifications of the *Betz-Joukowsky limit*) which should be removed now.

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