WEIGHTING FUNCTION SELECTION PROBLEM IN THE GALERKIN METHOD

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Summary When a problem contains more than one unknown function and a corresponding number of differential equations, the contents of the Galerkin method are not necessarily completely clear. This theme is discussed in the paper by considering as demonstration examples one-dimensional heat conduction and bending of Euler-Bernoulli beam. A strategy for proper weight function selection is proposed. Some numerical results obtained by the finite element method are presented.

INTRODUCTION

The Galerkin method is one version of the method of weighted-residuals. This topic is described elegantly in Reference [1]. With the ever growing importance of the finite element method, the proper understanding of the contents of the Galerkin method has become important in the teaching of computational mechanics for engineering students. Normally, the subject is explained in the basic textbooks in the case of only one unknown function and one differential equation. Then the situation seems to be rather clear. However, when the problem contains more than one unknown function and a corresponding number of differential equation, the contents of the Galerkin method are not necessarily obvious. This theme is considered in the present paper. One-dimensional heat conduction and bending of Euler-Bernoulli beam are employed as demonstration examples.

HEAT CONDUCTION

Equations

We consider as the first example the field equations

$$\frac{\mathrm{d}q}{\mathrm{d}x} - s \equiv q' - s = 0\,, \qquad q + k\frac{\mathrm{d}\theta}{\mathrm{d}x} \equiv q + k\theta' = 0\,, \tag{1}$$

valid in the domain $\Omega = (0, L)$. Here, the unknown functions are the heat flux $q = q(x)$ and the temperature $\theta = \theta(x)$. The given quantities are the heat source density $s = s(x)$ and the thermal conductivity $k = k(x)$. The first equation describes the energy balance and second is the constitutive relation according to the Fourier law. The boundary conditions are taken to be $q(L) = \bar{q}$ and $\theta(0) = \bar{\theta}$ with $\bar{q}$ and $\bar{\theta}$ as given.

Weak form

The conventional approach in heat conduction is to eliminate the heat flux in (1) to have a second order differential equation in temperature. However, here we keep q and θ as basic unknowns and obtain thus a very simple demonstration case with two unknowns and two differential equations.

The weak form corresponding to equations (1) is

$$\int_\Omega \left[w_1\left(q' - s\right) + w_2\left(q + k\theta'\right)\right] \mathrm{d}\Omega = 0\,, \tag{2}$$

where w_1 and w_2 are weighting functions. In the Galerkin method the weighting functions are taken to be the functions used in trial solution with undetermined parameters, [1]. However, this concerns the discrete formulation. Before the discretization is considered, the weighting functions in the Galerkin method can be interpreted as variations of the functions to be determined. Thus, we can write (2) as

$$\int_\Omega \left[\delta q\left(q' - s\right) + \delta\theta\left(q + k\theta'\right)\right] \mathrm{d}\Omega = 0\,. \tag{3}$$

But why have we associated δq with the first field equation and $\delta\theta$ with the second? Quite clearly, we can write equally well the alternative weak form

$$\int_\Omega \left[\delta\theta\left(q' - s\right) + \delta q\left(q + k\theta'\right)\right] \mathrm{d}\Omega = 0\,. \tag{4}$$

It should be noticed that a basic course on the finite element method can proceed without heavy use of variational calculus and the concept of a functional is perhaps not needed at all. Here, the variation can thus be defined simply as a short-hand notation for the difference between two admissible functions in the weak form. An admissible function is smooth enough and satisfies the possible Dirichlet boundary condition set on it. Thus the variation vanishes at a Dirichlet boundary.

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Discretization

C^0 - continuous finite element approximations are used to solve the above heat conduction problem. It is found that both the weak forms (3) and (4) produce satisfactory results but the latter formulation is more accurate.

SELECTION STRATEGY

Based on the experience with the numerical results with the heat conduction problem, there rises the question: can we have some in advance rules for making a "good" selection between the weighting functions? Modern programs, such as COMSOL Multiphysics [2] give the possibility to solve numerically rather arbitrary differential equation systems with many unknown functions. However, the applier must then generate an appropriate weak formulation and is thus confronted with the weighting function selection problem.
Let us write a possible linear differential system (in one dimension) in the form

$$\mathbf{Lu}-\mathbf{b} \equiv \mathbf{Cu}'+\mathbf{Ru}-\mathbf{b}=\mathbf{0}\,. \tag{5}$$

Here $\mathbf{u}$ is a column vector formed from the unknown functions, $\mathbf{C}$ and $\mathbf{R}$ are square matrices and the meaning of the rest of the notations is rather obvious. Here, for simplicity, only up to first order derivatives are assumed to be present. A corresponding weak form would be

$$\int_{\Omega}\mathbf{w}^{\mathrm{T}}\left(\mathbf{Lu}-\mathbf{b}\right)\mathrm{d}\Omega \equiv \int_{\Omega}\mathbf{w}^{\mathrm{T}}\left(\mathbf{Cu}'+\mathbf{Ru}-\mathbf{b}\right)\mathrm{d}\Omega=0\,. \tag{6}$$

We now assume the weighting function vector to be of the form

$$\mathbf{w}=\mathbf{P}\delta\mathbf{u}\,, \tag{7}$$

where $\mathbf{P}$ is a square matrix. The elements of $\mathbf{P}$ are originally unknown. We now demand that the modified system $\mathbf{P}^{\mathrm{T}}\mathbf{Lu}-\mathbf{P}^{\mathrm{T}}\mathbf{b}=\mathbf{0}$ following from the introduction of (7) becomes, if possible, self-adjoint. For reasons explained e.g. in [3], this is considered as beneficial. This leads to the conditions $\mathbf{P}^{\mathrm{T}}\mathbf{C}=-\mathbf{C}^{\mathrm{T}}\mathbf{P}$ and $\mathbf{P}^{\mathrm{T}}\mathbf{R}=\mathbf{R}^{\mathrm{T}}\mathbf{P}$. Additionally, $\mathbf{P}$ is assumed to be similar to the permutation matrix; however, the non-zero elements can be different from unity. In this way the non-zero elements of $\mathbf{P}$ can be hopefully determined. This procedure works in the heat conduction problem and produces with $\mathbf{u}=[\,q\,,\theta\,]^{\mathrm{T}}$ the selection $w_1=\delta\theta$ and $w_2=-1/k\cdot\delta q$ which corresponds — in a somewhat modified way — to formulation (4) and in fact leads to a symmetric coefficient matrix.

EULER-BERNOULLI BEAM

The second demonstration problem concerns the bending of an Euler-Bernoulli beam. The governing equations are

$$v'-\theta=0\,,\quad Q'+q=0\,,\quad Q-M'=0\,,\quad M+EI\theta=0 \tag{8}$$

in the domain $\Omega=(0,L)$. Here, v is the deflection, θ the cross-sectional rotation, M the bending moment and Q the shearing force. The given quantities are the loading per unit length q and the bending stiffness EI. The beam is assumed to simply supported at the left end and clamped at the right end so the boundary conditions used are $v(0)=0$, $M(0)=0$, $v(L)=0$, $\theta(L)=0$.
The conventional approach would consist of eliminations so that finally a fourth order differential equation for the deflection is obtained. Here the four unknowns are kept as such and now the weighting function selection problem appears in a more involved way than in the heat conduction problem. With $\mathbf{u}=[\,v\,,\theta\,,M\,,Q\,]^{\mathrm{T}}$, the selection strategy proposed above gives the weightings $w_1=-\delta Q$, $w_2=\delta v$, $w_3=\delta\theta$, $w_4=1/EI\cdot\delta M$. This is found to work well in connection with the finite element method. On the contrary, for instance, with the "blind" selection $w_1=\delta v$, $w_2=\delta\theta$, $w_3=\delta M$, $w_4=\delta Q$, the discrete solution gives wildly oscillating useless results.

CONLUSIONS

In teaching of the finite element method it is important to emphasize that the Galerkin method contains a selection problem with respect to the weighting functions. In some cases the strategy proposed in this paper may help in making a proper selection.

References

[1] Crandall S. H.: Engineering Analysis, A Survey of Numerical Procedures, McGraw-Hill, 1956.
[2] www.comsol.com.
[3] Lanczos C.: Linear Differential Operators, D. Van Nostrand, 1960.