EFFECTIVE METHODS OF TEACHING VECTORS IN UNIVERSITY INTRODUCTORY MATHEMATICS

Akinola, A. P.,[1] Layeni, O. P. & Olagunju, M. A.
Department of Mathematics,
Obafemi Awolowo University, Ile-Ife, Nigeria.

INTRODUCTION

The ways and means of making mathematics/mechanics learning less burdensome had been and will always be on the front burner of pedagogic engagement. To underscore this, you readily hear the word "*mathphobia*" (i.e. "phobia for mathematics") or "*tensor-phobia*" (i.e. "phobia for tensor"). This can be contextualized in the challenge posed by Professor G. M. Weinberg's statement on "*one hope of the General System Theorist*", which translates into "*One of the aims of the general systems theorists is to make ability or skill in a given area of human endeavour less dependent on genius*".

This could be interrogated from the point of view of the concept of formal cybernetics. In fact, Professor S. Beer, one of the acclaimed founding fathers of cybernetics, in his numerous works on the subject enunciated the procedures for making skill in certain areas of mathematical work less dependent on inborn ability or instinct [1]. Adeagbo-Sheikh [1] projected this concept to making learning less dependent on luck and went ahead to model a body of knowledge in mathematics for reading and understanding difficult texts. For this, he identified the concept of self-organizing systems and gave a general conceptual framework/model for studying this system. In the conceptual model the connectivity of the pools of knowledge was examined. The concept of ordering and topological property of connectedness was used to model the mathematical body of knowledge and to prove the workability of a procedure for reading and understanding difficult mathematics texts.

Really, most basic engineering designs make copious use of vector concepts [2]. A proper grasp of this is quite imperative. But more, any serious modern designs and engineering analysis, especially in continuum mechanics, is unimaginable without firm understanding of the concept of tensor [3,4,5]. In fact, in continuum mechanics without being well grounded in tensor it would be quite challenging to practically address some vital concepts: invariants of objects such as stress, strain, material characteristics and others that copiously make use of the Hamilton-Kelly theory; energy, energy function and its tensor derivative [4]; homogenization theory that enables the delicate navigation through complex relations involved in heterogeneous/composite media and the deduction of the effective characteristics of composites, from mechanics point of view [5]; and others. Invoking the concept of cognitive learning method and formal cybernetics this paper gives the opinion that the subject of vectors at the University level should be introduced to students from the point of view of *invariant object*, such that a vector is known as an invariant object rank-1, which naturally prompts and tasks the mind and provokes the curiosity of students to ask the question as to what constitutes an object of rank-0 or rank-2 or indeed rank-n, for any natural number n. In this context it becomes easy for students to see a scalar as a rank-0 object and helps to understand or avoid the ambiguity inherent in taking components of a vector as scalars, as often expressed in many texts.

Teaching experience over the years [6], revealed that while students show some degree of understanding of vector concepts they depict tendency towards "*tensor-phobia*". The development of appropriate cognitive teaching approach becomes imperative. Hence, the need to let students see early enough the connection between tensor and vectors, for which they already had acquaintance from the secondary school. This first impression carries over and lasts longer, making them carry out simple operations, which they were used to, such as scalar product, vector product and others on vectors, using *invariant/rank* concept. As an illustration, we consider *dot* and *cross* vector product operations using the *isotropic tensors*.

DOT AND CROSS VECTOR PRODUCT OPERATIONS AND ISOTROPIC TENSORS

[1] *Corresponding author. Email: aakinola@oauife.edu.ng; Currently, a Visiting Professor, Department of Mathematics, University of Ibadan, IBADAN, Nigeria*

Consider, without loss of generality, vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and the rank-2 *isotropic tensor* $\underset{\sim}{\mathbf{I}} = g^{ij}\mathbf{e}_i\mathbf{e}_j$ and rank-3 *isotropic tensor* $\underset{\sim}{\epsilon} = \epsilon^{ijk}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k$ in the three dimension Euclidean space $\Re^3$, $i, j, k = 1, 2, 3$; $\mathbf{e}_i$ are the orthogonal base vectors. Then recalling from the elementary vector theory, *dot product* is $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$, product is $\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$, while the *scalar triple product* is $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$. Now, the cross product of vectors $\mathbf{a}$ and $\mathbf{b}$ can be given with the aid of $\underset{\sim}{\epsilon}$, such that $\mathbf{a}\times\mathbf{b} = \underset{\sim}{\epsilon}\cdot\cdot\,\mathbf{a}\mathbf{b}$. In fact,

$$\epsilon^{ijk}a_ib_j\mathbf{e}_k = \mathbf{a}\times\mathbf{b} \equiv \begin{vmatrix} \mathbf{k}_1 & \mathbf{k}_2 & \mathbf{k}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{k}_1 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}\mathbf{k}_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}_3 = c_m\mathbf{k}_m \equiv \mathbf{c}, \tag{1}$$

where $m = 1, 2, 3$, c_1, c_2, c_3 are the minor to the orthonormal base vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ respectively; and for simplicity, in the cartesian coordinates, the components of $\underset{\sim}{\epsilon}$ become the *Levi Chivita symbols* $e^{ijk} = e_{ijk}$,

$$\epsilon_{ijk} \equiv \mathbf{e}_i\cdot(\mathbf{e}_j\times\mathbf{e}_k) = ve_{ijk};\; e_{ijk} \equiv \mathbf{k}_i\cdot(\mathbf{k}_j\times\mathbf{k}_k) = \begin{cases} 1 & \text{when } i \neq j \neq k \neq i,\ cp(ijk) \text{ is even}, \\ 0 & \text{when any of } i, j, k \text{ are equal}, \\ -1 & \text{when } i \neq j \neq k \neq i,\ cp(ijk) \text{ is odd}, \end{cases} \tag{2}$$

where $i, j, k = 1, 2, 3$ and $cp(ijk)$ means cyclic permutation of the indices i, j, k : v is the volume of parallelpiped built on the orthogonal basis vector $\mathbf{e}_i$. Also, we have the relationship between $\underset{\sim}{\mathbf{I}}$ and $\underset{\sim}{\epsilon}$

$$\underset{\sim}{\epsilon} = -\underset{\sim}{\mathbf{I}}\times\underset{\sim}{\mathbf{I}};\; \epsilon^{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix};\; \delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j, \end{cases} \quad i, j = 1, 2, 3,$$

where δ_{ij} is the so-called *Kronecker delta*. Further, given any rank-2 tensor $\underset{\sim}{\mathbf{T}}$, its *cofactor tensor* $\underset{\sim}{\mathbf{T}}^x$ can be obtained simply as the *dyad* of vector $\mathbf{e}_i\times\mathbf{e}_j$ and vector $(\underset{\sim}{\mathbf{T}}\cdot\mathbf{e}^i)\times(\underset{\sim}{\mathbf{T}}\cdot\mathbf{e}^i)$ such that

$$\underset{\sim}{\mathbf{T}}^x = [(\underset{\sim}{\mathbf{T}}\cdot\mathbf{e}^i)\times(\underset{\sim}{\mathbf{T}}\cdot\mathbf{e}^i)][\mathbf{e}_i\times\mathbf{e}_j] = \frac{1}{2}\epsilon_{skq}\epsilon^{mnp}T_n{}^kT_p{}^q\mathbf{e}^s\mathbf{e}_m.$$

Thus, the *1st invariant* $I_1(\underset{\sim}{\mathbf{T}})$ and *2nd invariant* $I_2(\underset{\sim}{\mathbf{T}})$ of tensor $\underset{\sim}{\mathbf{T}}$ can be given through the double dot product:

$$I_1(\underset{\sim}{\mathbf{T}}) = \underset{\sim}{\mathbf{I}}\cdot\cdot\,\underset{\sim}{\mathbf{T}},\; I_2(\underset{\sim}{\mathbf{T}}) = I_1(\underset{\sim}{\mathbf{T}}^x) = \underset{\sim}{\mathbf{I}}\cdot\cdot\,\underset{\sim}{\mathbf{T}}^x = \frac{1}{2}[I_1^2(\underset{\sim}{\mathbf{T}}) - I_1(\underset{\sim}{\mathbf{T}}^2)].$$

This process can be followed to compute other invariants, obtain the Hamilton-Kelly relation, tensor derivative of scalar functions and thus deduce constitutive relations from pertinent energy; essentially, using already known operations of dot and cross products on invariant objects, vectors inclusive.

References

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