NOISE SENSITIVITY OF BALANCE WITH DELAYED ON-OFF CONTROL

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Summary Models of balance have recently attracted considerable attention in mechanics and robotics, as well as biology. Through a canonical model of balancing we consider the influence of different noise sources on act-and-wait control. Measures of variation in stability and optimal control due to randomness are given through both numerical and analytical methods. Comparison is given for the case of feedback control with noise, to contrast the influence of noise in the different systems.

MODELS OF BALANCE IN APPLICATIONS

Motion of an inverted pendulum with delayed feedback is a classical model in human balancing tasks and mechanical systems (see, e.g.[1]-[4]). Time-delayed PD control (with the applied forces proportional to the position (P) and its derivative (D)) has been used in models of balance with a controllable cart [1,4] and torque controlled balance as in human postural sway [3]. Time-delay is fundamental in a variety of balancing problems for capturing the time-lag of measurements and between applied forces, observed in both human balance and mechanical and electronic systems. Recently it has been recognized that the realistic or ideal control may be one that is switched on and off, as proposed to reflect observations of intermittent muscle movements, to achieve simplicity in mechanical systems, or to provide a stabilizing mechanisms when the time-delay is long [2][5]. The presence of delayed feedback, nonlinearities, and on-off control all contribute to the complexity of analyzing these systems, as well as to sensitivity to random effects. Here we concentrate on two types of on-off control: act-and-wait control and feedback (or state-dependent) control, described further below in the context of the model. We are interested in understanding the influence of different types of random fluctuations on the stability of the balanced state.

THE CANONICAL MODEL

Models of balance in mechanical and biology have a basic structure of that of an inverted pendulum controlled by either pivot (torque) or moveable base (cart) control. A non-dimensionalized model that captures the main effects of balancing in a variety of nonlinear systems, further streamlined with an appropriate time scaling, is given by

$$\ddot{\theta} - \sin\theta = F\cos\theta, \quad F(t) = -D\dot{\theta}(t-\tau) - P\theta(t-\tau) \qquad (1),$$

where θ is a dimensionless quantity representing the angular displacement of the inverted pendulum from vertical and F denotes control of the balancing actions. The control force F is considered in the simplest form of a PD controller with constant gains P and D chosen by the operator appropriately. The parameter τ gives the time of delayed feedback in the system. While model (1) is simple, it nevertheless captures the main features of the effects of the nonlinearity and delay that is observed for human balance [3] as well as those of an inverted pendulum controlled by the movement of a controllable cart in the limit when the mass of the pendulum is small compared with that of the cart [6].

In [5], the act-and-wait control was applied to motion of a block mass, showing the efficiency of the act-and-wait control that yields a larger stabilizing region of control parameters. Act-and-wait control is characterized by cycling between a sampling period with feedback gains due to application of the control (act), followed by no control for a certain number of periods (wait). We contrast the effects of noise in system (1) with act-and-wait control for two different noise sources, when P and D are random variables, and from additive noise given by the system,

$$\ddot{\theta} - \sin\theta = F\cos\theta + \eta(t) \qquad (2)$$

for η white noise. We also compare to a feedback control system, as given in [3] and analyzed in [6], with control applied under the following condition,

$$\theta(t-\tau)(\phi(t-\tau) - s\theta(t-\tau)) > 0 \qquad (3)$$

Here the parameter s<0, typically smaller than unity, defines a switching manifold in the phase plane (See Fig. 1, right panel). The switching manifolds, denoted Σ_1, Σ_2 in the phase plane, effectively divide the dynamics into an "ON" region where the control is on and the system would drift away from the θ=0 without it, and an "OFF" region where the system drifts towards the origin without the control.

RESULTS FOR ACT-AND-WAIT CONTROL

We first analyze the deterministic system with act-and-wait control, linearized about the equilibrium $\theta=0$, to find the optimal control values $P=P^*$ and $D=D^*$ using the monodromy matrix approach developed in [5]. Then we find a larger parametric region for stability of the desired equilibrium of $\theta=0$, with the potential for deadbeat control for optimal values. Deadbeat control is achieved for P^*, D^* in an act-and-wait cycle of control that is "on" for Δt and "off" for $m^*\Delta t$,, m=n or larger, with the total act-and-wait cycle of length $(n+1)^*\Delta t$. For P and D randomly distributed about P^*, D^*, respectively, we observe similar behavior for a range of parameters, and report here the results for the

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parameter values for the sampling period Δt = .04 for the active control and a delay period of $\tau = n^*\Delta t$, n=5. The noise sensitivity is illustrated in the left panel of Figure 1 with variation of the decay of θ to 0. These fluctuations differ from the deterministic system with optimal (deadbeat) control parameters P= P* and D=D* that yields a nearly immediate transition to θ=0 once the control is on. Underlying the fluctuation in they noisy system is the variation of the eigenvalue for the system. The calculation of the distribution of the eigenvalue for the system with random *P* and *D* shows significant probability for the eigenvalues taking values away from zero. Consideration of the probability density of the eigenvalue shows consistency with the observation that the rapid fluctuation of the random control parameters yields a faster rate of decay to θ = 0: the system then often has the opportunity to move away from the larger eigenvalues. Fig 1 gives a comparison with additive (external) noise, where larger excursions from the balanced state can be observed, providing a larger probability of tipping in the nonlinear setting where other states are possible.

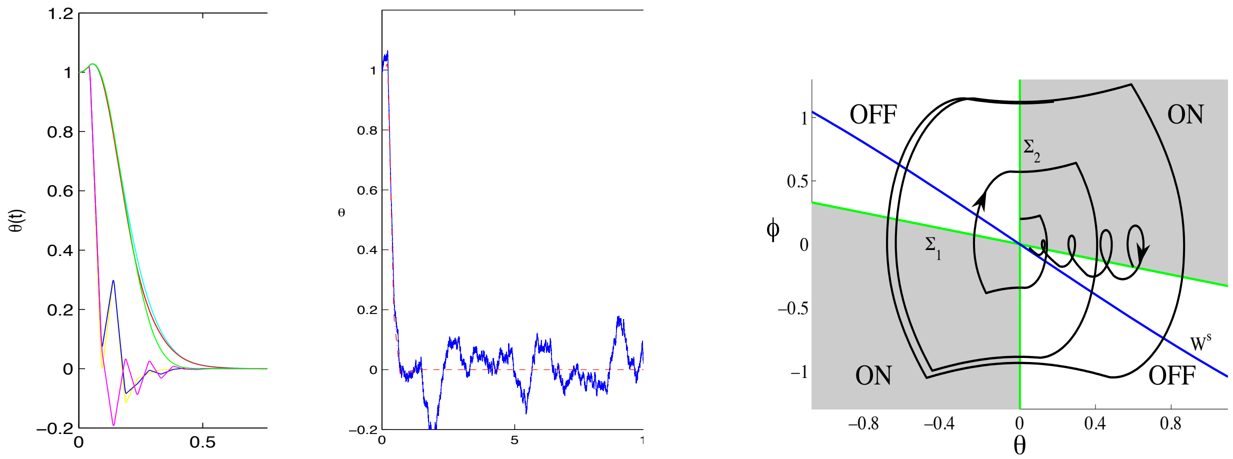

Figure 1: Left: Simulations of system (1) with random P and D. The smooth curves correspond to control always on, the jagged curves correspond to act-and-wait control. The time to reach the equilibrium for random P and D is roughly 4-5 times that of act-and-wait control for deterministic deadbeat control values P=P*, D=D*. Centre: Simulations of (2), additive noise with act-and-wait control. Right: Phase-plane for θ, $\phi = \dot{\theta}$ for feedback control, on-off regions determined by (3).

RESULTS FOR FEEDBACK CONTROL

In the case of feedback control, we have analyzed bifurcations and phase plane dynamics in the setting described by 1), 3), and shown in Figure 1, right panel [6]. The transients and stability are affected by the combination of time-delay, discontinuity in the control, and nonlinearity. There are two key situations where the system is sensitive to noise: First, when there are periodic solutions that are bistable with the $\theta=0$ equilibrium, small noise can drive transitions away from the stable inverted pendulum to a swaying periodic solution. Second, there are noise sensitivities for certain combinations of control parameters and delay driving dynamics that approach the line W^s, the stable manifold of the origin for the OFF system. In this second case small noise can cause a drastic shift from attraction to a zig-zag behavior where the system approaches the origin from one side while switching between the ON and OFF (see Fig. 1) to that of a spiral type behavior, where the system circles the origin repeatedly, without approaching the origin. Presence of the noise causes a repeated transition to the spiraling behavior rather than an approach to the origin via a zig-zag behavior. Examples of zig-zags and spirals are shown in Fig. 1, right panel.

CONCLUSIONS

We compare and contrast the effect of noise in different types of on-off control in balance models, providing complementary analysis and computations to better understand the sensitivities of these models to different random sources. Systems with act-and-wait control experience noise sensitivities due to variation of the eigenvalues of the underlying linearized systems, and are sensitive to external fluctuations in the "waiting" intervals. Systems with feedback control exhibit sensitivities to randomness when nonlinearities, delays, and on-off control support potential transitions between co-existing behaviors.

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