CLASSICAL AND COSSERAT ELASTO-PLASTIC CONSTITUTIVE EQUATIONS FOR PRESSURE-DEPENDENT YIELD

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Summary This paper presents the equations of a novel elasto-plastic model for the deformation and flow of granular materials. The model overcomes the well-known shortcomings of the standard plastic potential and double-shearing models, including that fact that these models do not depend continuously on the initial data for evolutionary problems. A model for a classical continuum is embedded into a heirarchy of Cosserat models and observations are made concerning the development of Cosserat models.

INTRODUCTION

That the continuum description of the deformation and flow of granular materials has proven a remarkably difficult problem to solve may not come as a surprise given the discrete nature of granular systems. Nevertheless, it *is* a remarkable fact that no theoretical description has found universal acceptance for systems comprising sufficiently large volumes of dense packings of sufficiently small grains. The plastic potential model [1] for pressure dependent yield, when generalised to a non-associated flow rule, may be regarded as incomplete. It may be completed by the addition of a term which plays an analogous role to the so-called non-coaxial term in the double-shearing model [2]. In this paper we observe that the classical continuum is insufficient in regard to both kinematic and stress quantities to adequately formulate and describe deformation and flow of granular materials. It is contended that a Cosserat continuum does sufficiently describe such flows but that in most of the volume occupied by the material the Cosserat effects are negligible, the exception being in shear bands and in the immediate vicinity of boundary surfaces comprising elastic or rigid material. Finally, it is contended that the equations governing steady state plane or anti-plane deformation should exhibit spatial hyperbolicity, evolutionary elasto-plasticity equations should be hyperbolic and evolutionary rigid-plastic equations parabolic. This is the ideal mathematical framework for plasticity theory and should be adhered to if at all possible. The model here develops and generalises those presented in [3,4].

MATHEMATICAL NOTATION AND FORMULATION

Let Ox_i, $i = 1, 2, 3$ denote rectangular Cartesian coordinate axes and let $\mathbf{x} = (x_i)$ denote the position vector of an arbitrary point P. Relative to an Eulerian formulation let $\mathbf{v} = (v_i)$ denote the velocity field and $\sigma = (\sigma_{ij})$, $j = 1, 2, 3$, denote the Cauchy stress tensor. Let $\mathbf{\Gamma} = (\Gamma_{ij}) = (\partial v_i/\partial x_j)$ denote the velocity gradient tensor and denote the symmetric and anti-symmetric parts of $\mathbf{\Gamma}$ by $\mathbf{d}$ and $\mathbf{s}$ and call them the *velocity strain* and *velocity spin*, respectively

An extra kinematic quantity is required to formulate the constitutive behaviour even in cases where, with regard to magnitude, Cosserat quantities are negligible. This quantity is called the (vector) intrinsic spin $\mathbf{\Omega}^v = (\Omega_i)$. Both $\mathbf{v}$ and $\mathbf{\Omega}$ are primitive quantities in the continuum models and do not require definition in terms of an underlying discrete model of the system. However, intuitively we may envisage a Representative Volume Element (RVE) surrounding a point P containing several grains and regard $\mathbf{v}$ and $\mathbf{\Omega}$ as the mean translational momentum per unit mass and the mean rotational angular momentum per unit moment of inertia of the RVE, respectively. One of the fundamental kinematic quantities required to describe granular materials is the *relative spin*, denoted by ω, namely the velocity spin relative to the intrinsic spin,

$$\omega = \mathbf{s} - \mathbf{\Omega} \tag{1}$$

where $\mathbf{\Omega}$ denotes the second order anti-symmetric dual tensor of $\mathbf{\Omega}^v$. Since the relative spin is frame-indifferent (being the *difference* of two spins) it may be used in constitutive equations. In a classical continuum we shall define the intrinsic spin to be a constant vector (the constant may always be taken to be zero by a suitable choice of reference frame and, further, this frame is usually at rest relative to the (non-rotating) boundaries of the material).

CLASSICAL CONTINUUM: CONSTITUTIVE EQUATIONS

The existence is assumed of a function f satisfying at point P the conditions of *yield* and, while the state of yield continues in time, of *continuing yield*

$$f(\sigma) \leq 0, \qquad \mathbf{f} \cdot \dot{\sigma} = 0, \tag{2}$$

where the superposed dot denotes the material derivative and the second order tensor $\mathbf{f} = (f_{ij}) = (\partial f/\partial \sigma_{ij})$ may be called the *yield moduli*. The elasto-plastic type flow rule is given by

$$\mathbf{d} = \mathbf{M}^s : \mathring{\sigma}^s + \dot{\lambda}\mathbf{g}^s + \mu\,(\omega \cdot \sigma^s - \sigma^s \cdot \omega)\,, \qquad d_{ij} = M_{ijkl}\mathring{\sigma}^s_{kl} + \dot{\lambda}g^s_{ij} + \mu\left(\omega_{ik}\sigma^s_{kj} - \sigma^s_{ik}\omega_{kj}\right), \tag{3}$$

where $\mathbf{M}^s$ denotes the elastic compliance tensor, the superposed $^\circ$ denotes an objective rate (usually the Jaumann-rate), superscript ^{s} merely emphasises that the tensor is symmetric, $\dot{\lambda}$ denotes an arbitrary non-negative scalar multiplier, $\mathbf{g}$

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denotes a second order tensor called the tensor of *flow moduli* and μ denotes a scalar function. $\mathbf{g}$ is said to admit the *plastic potential* g in the case where there exists a scalar function $g(\sigma)$ such that $\mathbf{g} = (g_{ij}) = (\partial g/\partial\sigma_{ij})$. When the system of equations comprising the stress equations of motion, continuing yield and flow rule are hyperbolic and the characteristics uncouple into stress and velocity characteristics (this is certainly the case for all deformations of Mohr-Coulomb materials and for all general plane and anti-plane deformations) then the scalar function μ is determined by the condition that the stress and velocity characteristics coincide. Thus, neither $\dot{\lambda}$ nor μ are material constants, $\dot{\lambda}$ is arbitrary (but non-negative) while μ is determined uniquely by $\mathbf{f}$ and $\mathbf{g}$. Many deformations and flows can be adequately described by the above constitutive equations together with the stress equations of motion. This may fail to be true in shear bands and in the vicinity of solid bounding surfaces. In cases where it is necessary to consider the shear-band or boundary layer to have finite width or thickness and to describe the flow within the shear band or surface boundary layer then a Cosserat model needs to be adopted, i.e. the evolution and value of the intrinsic spin Ω must be computed. Equations and quantities (with the exception of intrinsic spin) governing the classical continuum model will be referred to as translational equations and quantities in the sequel.

COSSERAT CONTINUUM: CONSTITUTIVE EQUATIONS

In developing a Cosserat model it is convenient to decompose the deformation into translational and rotational parts (although it should be noted that the translational equations of motion do not decompose in this way). This can be done without loss of generality when equation (3) is used but does involve loss of generality if this principle is extended to the yield condition. It should be noted that the more interconnections there are between the Cosserat and non-Cosserat variables the more likely it is that they will interact in an inconsistent way and cause ill-posedness. It is thus desirable to assume the simplest possible connections between the two sets of variables to obtain a well-posed model. We regard this as an application of Occam's razor which does not contradict the Principle of Equipresence since the latter merely requires us to consider the possibility that all possible connections between variables may be present, it does not require us to actually assume that they are all present. In a similar way, Cosserat quantities may be generally assumed "small" in comparison to translational quantities (for otherwise Cosserat effects would be widely and unambiguously observed). Hence we should not give equal status to equations governing rotational quantities and to equations governing translational quantities. In the first instance, it is sufficient to take the simplest possible Cosserat model. A simple Cosserat equation should only be replaced by a more complicated equation in response to some definite piece of experimental or theoretical evidence.

Let μ, with components μ_{ij}, denote the couple stress. In the presence of couple stress the Cauchy stress need not be symmetric; let σ^s and σ^a denote the symmetric and anti-symmetric parts of σ. Equation (3) still holds in the Cosserat model, except now the superscript ^{s} denotes the symmetric part of the Cauchy stress.

It is usual to assume a single yield condition

$$f(\sigma, \mu) \leq 0, \tag{4}$$

but it may be possible to decompose yield into symmetric and anti-symmetric parts

$$f^s(\sigma^s) \leq 0, \qquad f^a(\sigma^a, \mu) \leq 0, \tag{5}$$

or even into three parts

$$f^s(\sigma^s) \leq 0, \qquad f^a(\sigma^a) \leq 0, \qquad f^\mu(\mu) \leq 0, \tag{6}$$

if it simplifies the model but describes the flow with sufficient accuracy. The advantage of doing this is that the less coupling there is between Cosserat and non-Cosserat quantities the more the properties of the Cosserat model resemble those of the classical model and will hence be more likely to preserve any good properties that the latter may possess.

Without loss of generality, the companion equation to equation (3) for σ^a when equation (4) holds is

$$\omega = \dot{\lambda}\frac{\partial g^a}{\partial \sigma^a}. \tag{7}$$

In the case where equations (5) or (6) hold $\dot{\lambda}$ is replaced by a new arbitrary multiplier $\dot{\lambda}^a$ and g^a may be replaced by f^a. The constitutive equation for the couple stress may be written in terms of an associated flow rule

$$\nabla\mathbf{\Omega} = \dot{\lambda}^a\frac{\partial f^a}{\partial \mu} \qquad \text{or} \qquad \nabla\mathbf{\Omega} = \dot{\lambda}^\mu\frac{\partial f^\mu}{\partial \mu}. \tag{8}$$

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