EDELEN'S DISSIPATION POTENTIALS AND THE VISCOPLASTICITY OF GRANULAR MEDIA

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Summary In remarkable works largely overlooked in continuum mechanics, D.G.B. Edelen (1972-74) presents a purely mathematical construction of dissipation potentials, as an extension of the classical Rayleigh-Onsager theory to non-linear dissipative systems characterized by n-dimensional force $\mathbf{x}$ conjugate flux $\mathbf{y}$ and positive dissipation $D = \mathbf{y} \cdot \mathbf{x}$. Edelen's formulae allow construction of complementary potentials $\varphi(\mathbf{x})$ and $\psi(\mathbf{y})$ from given $\mathbf{y}(\mathbf{x})$ or $\mathbf{x}(\mathbf{y})$, with forces and fluxes given respectively by gradients of these potentials, modulo dissipation-free terms also given by Edelen. This paper summarizes several applications to the non-polar and polar history-dependent visco-plasticity of granular media, and it discusses the relation to phenomenological models based on certain thermodynamic postulates.

Introduction

As an extension of the classical quadratic form of Rayleigh and Onsager, Edelen's mathematical construct of a more general *dissipation potential* [2, 3, 4] applies to any strictly dissipative process, linear or non-linear. Familiar physical examples are provided by discrete networks of non-linear electrical resistors, by systems of chemical reactions, and by various continuum models of plasticity [13, 10, 1, 7] or viscosity [6].

The existence and derivation of dissipation potentials are often predicated on extremum principles, such as maximal (or extremal) dissipated power and entropy production [13] or on related thermodynamic postulates, well summarized by the comprehensive work of Collins & Houlsby [1]. Otherwise, they are assumed as a matter of convenience, as apparently done in the treatise of Hill [8], or, finally, they taken as given in mathematical works such as that of Moreau [12]. By contrast, Edelen's purely mathematical derivation of such potentials from given force-velocity-dissipation relations is virtually free of special assumptions and, as such, it merits much more attention than it has apparently received, particularly in the field of continuum mechanics.

Not only is the existence of a dissipation potential of theoretical interest in its own right, but it is also essential to certain long-standing variational principles of continuum mechanics.

Edelen forms and transforms

In conventional vector notation, Edelen's [3] generalization of the Onsager relations is:

$$\mathbf{y}(\mathbf{x}) = \nabla_{\mathbf{x}}\varphi(\mathbf{x}) + \mathbf{u}(\mathbf{x}), \text{ with } \mathbf{x}\cdot\mathbf{u} = 0, \text{ with } \mathfrak{D} = \mathbf{y}\cdot\mathbf{x} = \mathbf{x}\cdot\nabla\varphi \geq 0, \tag{1}$$

where $\mathbf{x} \in \mathbb{X} \cong \mathbb{R}^n$ represents force, dual to to conjugate flux $\mathbf{y} \in \mathbb{X}^* \cong \mathbb{R}^n$, $\mathfrak{D}$ is dissipated power, φ is dissipation potential, and $r = ||\mathbf{x}||$ is an appropriate norm for force. Edelen's relation between the convex dissipation potential $\varphi(\mathbf{x})$ and convex dissipation function $\mathfrak{D} = D(\mathbf{x})$ can be expressed as the linear transforms:

$$\varphi(\mathbf{x}) = \mathfrak{E}\{D\} := \int_0^1 D(\lambda\mathbf{x})\frac{d\lambda}{\lambda}, \text{ with } D(\mathbf{x}) = \mathfrak{E}^{-1}\{\varphi\} = \mathbf{x}\cdot\nabla\varphi(\mathbf{x}) = r\frac{\partial\varphi}{\partial r}, \tag{2}$$

provided that $D(\mathbf{x}) = o(r^\nu)$ with $\nu > 0$, for $r \to 0$. If $\{D, \varphi\}$ are homogeneous of degree-q, such that $D(\mathbf{x}) \equiv q\varphi(\mathbf{x})$, then φ, q are an eigenfunction-eigenvalue pair for $\mathfrak{E}^{-1}$, which allows for the corresponding spectral representation of $\mathfrak{E}^{-1}$ in terms of homogenous functions. Examples are are the multivariate Taylor series expansion in $\mathbf{x}$ for C^∞ functions, involving a discrete spectrum, and the multivariate (inverse) Mellin transform, involving a continuous spectrum. Thus, one may designate as spectral representations the expansions in polynomial invariants, such as those employed by Ziegler [14] for Reiner-Rivlin fluids, special forms of which have also been proposed as somewhat oversimplified models of granular mechanics [5, 9].

Not recorded here is Edelen's [3] formula for the non-dissipative "gyroscopic" term $\mathbf{u}$ in (1) in terms of flux $\mathbf{y} = \boldsymbol{\eta}(\mathbf{x})$.

Convexity, duality and dissipative moduli

The roles of force and flux may be exchanged, with Legendre-Fenchel duality $\varphi(\mathbf{x}) + \psi(\mathbf{y}) = \mathbf{y}\cdot\mathbf{x}$ between convex conjugate potentials $\varphi,\ \mathbb{X} \to \mathbb{R}$, and $\psi,\ \mathbb{X}^* \to \mathbb{R}$. where φ and ψ are assumed to be convex and differentiable, with invertible map $\mathbb{X} \rightleftharpoons \mathbb{X}^*$ arising from dual extrema: $\psi(\mathbf{y}) = \sup_{\mathbf{x}}\ \mathbf{y}\cdot\mathbf{x} - \varphi(\mathbf{x})$, and $\varphi(\mathbf{x}) = \sup_{\mathbf{y}}\ \mathbf{y}\cdot\mathbf{x} - \psi(\mathbf{y})$.

The dual of (2) gives $\psi(\mathbf{y})$ in terms of $\mathfrak{D} = D^*(\mathbf{y})$, and differentiability leads to *pseudo-linear* forms:

$$\begin{aligned}
&\mathbf{y} = \boldsymbol{\eta}(\mathbf{x}) = \nabla_{\mathbf{x}}\varphi = \mathbf{L}\mathbf{x}, \text{ and } \mathbf{x} = \boldsymbol{\xi}(\mathbf{y}) = \nabla_{\mathbf{y}}\psi = \mathbf{R}\mathbf{y},\\
&\text{with } \mathbf{L}(\mathbf{x}) = \nabla_{\mathbf{x}}\nabla_{\mathbf{x}}\psi((\boldsymbol{\eta}(\mathbf{x})), \ \mathbf{R}(\mathbf{y}) = \nabla_{\mathbf{y}}\nabla_{\mathbf{y}}\varphi((\boldsymbol{\xi}(\mathbf{y})) = \mathbf{L}^{-1},\\
&\text{and } \mathfrak{D} = D(\mathbf{x}) = \mathbf{x}\cdot\mathbf{L}\mathbf{x} = L_{ij}x^ix^j = D^*(\mathbf{y}) = \mathbf{y}\cdot\mathbf{R}\mathbf{y} = R^{ij}y_iy_j
\end{aligned} \tag{3}$$

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with symmetric positive moduli (*conductance* and *resistance*) $\mathbf{L}$ and $\mathbf{R} = \mathbf{L}^{-1}$. This represents a weak extension of Edelen's [3] non-linear generalization of Onsager's relations. The first two relations in (3) validate, generalize and supersede the "pseudo-linear" forms proposed previously [5, 7] for "purely dissipative" continua endowed with symmetric positive visco-plastic modulus.

The textbook example of convex conjugates involving homogeneous functions $D(\mathbf{x})$ and $D^*(\mathbf{y})$ of respective degrees $p > 1$ and $q > 1$ involves several noteworthy special cases.

For $p = q = 2$, one obtains quadratic forms representing to Rayleigh-Onsager dissipation with linear force-rate relations, akin to the model proposed by Lippmann [10] for Cosserat plasticity and employed by several researchers, e.g. [11], as a model of granular materials with length-scale effects. The marginal and singular limit $q \to 1, \ \ p \to \infty$, appropriate to rate-independent friction or plasticity,

implies the existence of limit (hyper)surfaces in $\mathbb{X}$ or $\mathbb{X}^*$. Hence, the existence of a plastic yield surface may be viewed as a mathematical consequence of rate independence and convex duality, point made by Collins & Houlsby [1] who trace its history. The "purely dissipative" model of granular materials and particle suspensions proposed by the present author [5, 6], and the related model of Jop et al. [9], are also subsumed by Edelen's theory.

Extremum principles and Conclusions

The preceding formulae serve to clarify the status of the various extremum principles that have been developed by Ziegler [13] and coworkers. In particular, it can be shown that, while results based on maximum (or stationary) dissipation give the correct form for the dissipative component of force or flux, the same is also true of an infinitude of similar force-flux relations.

The formulae proposed by Edelen serve to unify the description of a wide range of dissipative processes, including the viscoplasticity of granular media. The present work has attempted to provide a concise and critical overview. It leads to the conclusion that the prediction of dissipative force (e.g. stress) or flux (e.g. viscoplastic deformation rate), does not serve to elevate principles of stationary dissipation from any number of other models. In any event and in light of Edelen's work, such principles do not seem to offer overwhelming mathematical advantage in the derivation of a dissipation potentials from a given dissipation function. One point not addressed by the present work is the physical significance of Edelen's non-dissipative or "gyroscopic" force or flux, which may possibly be related to non-dissipative couplings or kinematic constraints such as granular dilatancy.

References

[1] I. F. Collins and G. T. Houlsby. Application of thermomechanical principles to the modelling of geotechnical materials. *Proc. Roy. Soc. A*, 453(1964):1975–2001, 1997.

[2] D. G. B. Edelen. A nonlinear Onsager theory of irreversibility. *Int. J. Eng. Sci.*, 10(6):481–90, 1972.

[3] D. G. B. Edelen. On the existence of symmetry relations and dissipation potentials. *Arch. Rational Mech. Anal.*, 51:218–27, 1973.

[4] D. G. B. Edelen. Primitive thermodynamics: a new look at the Clausius-Duhem inequality. *Int. J. Eng. Sci.*, 12(2):121–41, 1974.

[5] J. Goddard. Dissipative materials as constitutive models for granular media. *Acta Mech.*, 63(1):3–13, 1986.

[6] J. D. Goddard. A dissipative anisotropic fluid model for non-colloidal particle dispersions. *J. Fluid Mech.*, 568:1–17, 2006.

[7] J. D. Goddard. Granular hypoplasticity with Cosserat effects. In *Mathematical Modeling and Physical Instances of Granular Flows*, AIP Conference Proceedings V. 1227, pages 323–32. American Institute of Physics; New York, 2010.

[8] R. Hill. *The mathematical theory of plasticity*. Clarendon Press, Oxford, 1950.

[9] P. Jop, Y. Forterre, and O. Pouliquen. A constitutive law for dense granular flows. *Nature*, 441(7094):727–30, 2006.

[10] H. Lippmann. Eine Cosserat-Theorie des plastischen Fließens. *Acta Mech.*, 8(3):255–284, 1969.

[11] S.L. Mohan, K.K. Rao, and P. R. Nott. A frictional cosserat model for the slow shearing of granular materials. *J. Fluid Mech.*, 457:377–409, 2002.

[12] J. J. Moreau. Sur les lois de frottement, de plasticité et de viscosité. *Comptes Rendus*, 271:608–11, 1970.

[13] H. Ziegler. *An introduction to thermomechanics*, volume v. 21 of *North-Holland series in applied mathematics and mechanics*. Elsevier Science, Amsterdam; New York, 1983.

[14] H. Ziegler and L. K. Yu. Incompressible Reiner-Rivlin fluids obeying the orthogonality condition. *Arch. Appl. Mech.*, 41(2):89–99, 1972.