THICKNESS-SHEAR VIBRATIONS OF RECTANGULAR PLATES WITH FREE EDGES BY THE EXTENDED KANTOROVICH METHOD

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Summary The fundamental thickness-shear vibration frequency and mode shapes of rectangular plates of anisotropic materials with four free edges are of great interests in the analysis and design of quartz crystal resonators. We found that the extended Kantorovich method is the one with simple solution procedure and accurate results for this vibration problem. We compared approximate solutions with numerical results from finite element method and the agreement is satisfactory. This procedure is the improvement of earlier approaches and the difficulties in the accommodation of free boundary conditions have been overcome with the presented approximation technique.

INTRODUCTION

Quartz crystal plates in circular and rectangular configurations are core elements of piezoelectric resonators widely used in electronic circuits for frequency control and detection applications with the proliferation of sensors and wireless devices. In the design of thickness type resonators, the vibration frequency and mode shapes at the functioning mode, which is the thickness-shear mode in this study, are important information in the selection of resonator structures. Extensive efforts have been made in the analysis of high frequency vibrations of thickness-shear mode, which has a frequency much higher than the usual flexural vibrations we are familiar with. Due to the absence of accurate solutions of anisotropic rectangular plates with free edges, traditional approach is to use the one-dimensional solutions which neglected constraints of the width direction, which is called straight-crested solutions, as an approximation. Such solutions are generally acceptable and useful due to its simple procedure and good approximation based on experimental validation and numerical verification with the finite element method. Indeed, it has been confirmed that straight-crested waves exists in rectangular plates at the thickness-shear mode, and effects of the width direction on vibration frequency and mode shapes are negligible. Accurate results can only be obtained with the finite element method and finer mesh due to the high frequency and short wave length in the thickness and plane. On the other hand, methodologically and practically, analytical solutions with good accuracy are always sought for better determination of resonator structural parameters and estimation of performance. One example is the recent solutions of the frequency expression of thickness-shear vibrations based on the three-dimensional theory of piezoelectricity in agreement with one-dimensional solutions [1]. Analytical approaches proposed for solutions of rectangular Mindlin plates includes superimposition method [2] and the differential quadrature method [3], but the formulation and solution procedures are generally complicated and requiring extensive programming and evaluation process.

As an alternative approach based on our earlier research on vibrations of Mindlin plates with various boundary conditions, we believe the problem can be effectively solved by utilizing our earlier knowledge on vibration modes for approximate solutions based on the extended Kantorovich method. In this case, the mode shapes are formed with solutions of each direction as the initial guess, and functions are updated with new solutions. The boundary conditions of traction-free edges are approximated by enabling the work of stress components to vanish instead. Such a procedure has been effective in vibrations of rectangular Mindlin plates with various boundary conditions at lower frequencies. This study also shows the method is equally effective for the thickness-shear vibrations with free edges except the convergence is slower and mode shape functions need carefully chosen.

THICKNESS-SHEAR VIBRATIONS OF RECTANGULAR MINDLIN PLATES

The total potential functional

Consider a rectangular crystal plate of length $2a$, width $2b$ and uniform thickness h posses total potential functional

$$\begin{aligned} U &= \frac{1}{2}\int_{-b}^{b}\int_{-a}^{a}\left[D_1\left(\frac{\partial\phi}{\partial x}\right)^2 + 2D_2\left(\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial y}\right) + D_3\left(\frac{\partial\psi}{\partial y}\right)^2 + D_5\left(\frac{\partial\phi}{\partial y}+\frac{\partial\psi}{\partial x}\right)^2\right]\mathrm{d}x\mathrm{d}y \\ &+ \frac{1}{2}\int_{-b}^{b}\int_{-a}^{a}\left[D_6\left(\frac{\partial w}{\partial x}-\phi\right)^2 + D_4\left(\frac{\partial w}{\partial y}-\psi\right)^2\right]\mathrm{d}x\mathrm{d}y - \frac{\omega^2\rho}{2}\int_{-b}^{b}\int_{-a}^{a}\left(hw^2 + J\phi^2 + J\psi^2\right)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{1}$$

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where $J = h^3/12$, w is the flexure (deflection) and ϕ (ψ) are the thickness-shear (rotation) of plate, and

$$D_1 = \frac{h^3}{12}\left(C_{11} - \frac{C_{12}^2}{C_{22}}\right),\quad D_2 = \frac{h^3}{12}\left(C_{13} - \frac{C_{12}C_{23}}{C_{22}}\right),\quad D_3 = \frac{h^3}{12}\left(C_{33} - \frac{C_{23}^2}{C_{22}}\right)$$
$$D_4 = \frac{\pi^2 h}{12}\left(C_{44} - \frac{C_{24}^2}{C_{22}}\right),\quad D_5 = \frac{h^3}{12}C_{55},\quad D_6 = \frac{\pi^2 h}{12}C_{66} \tag{2}$$

in which $C_{ij}(i, j = 1, 2, 3, 4, 5, 6)$ are elastic constants referred to x, y, and z axes of the plate [4]. The stress resultants of the crystal plate are

$$M_x = -\left(D_1\frac{\partial\phi}{\partial x} + D_2\frac{\partial\psi}{\partial y}\right),\ M_y = -\left(D_2\frac{\partial\phi}{\partial x} + D_3\frac{\partial\psi}{\partial y}\right),\ M_{xy} = -D_5\left(\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial x}\right) \tag{3a}$$

$$Q_x = D_6\left(\frac{\partial w}{\partial x} - \phi\right),\ Q_y = D_4\left(\frac{\partial w}{\partial y} - \psi\right) \tag{3b}$$

The fundamental thickness-shear frequency has a magnitude of

$$\omega_0 = \frac{\pi}{h}\sqrt{\frac{C_{66}}{\rho}} \tag{4}$$

This study is focused on finding semi-analytical solutions using Eq. (1) in the vicinity of thickness-shear frequency in Eq. (4). For our problem of four free edges, we require the stress components to vanish as the standard boundary condition.

Solution Procedure of the Extended Kantorovich Method

According to the Kantorovich method, the flexure (deflection) w and thickness-shear (rotation) ϕ and ψ can be separated into two uncoupled function as

$$w = w_x(x)w_y(y),\quad \phi = \phi_x(x)\phi_y(y),\quad \psi = \psi_x(x)\psi_y(y) \tag{5}$$

First, we assume w_x, ϕ_x, ψ_x (or w_y, ϕ_y, ψ_y) as known functions and substitute Eq. (5) into Eq. (1). Applying the variational principle after integration, we can get a set of ODEs. The boundary conditions are satisfied by forcing the virtual work of stress resultants on boundaries to vanish. Solving the ODEs gives the first approximation of w_y, ϕ_y, ψ_y. Then we substitute Eq. (5) into Eq. (1) with the obtained w_y, ϕ_y, ψ_y. We can obtain the first approximation of w_x, ϕ_x, ψ_x using similar procedure as before. Now it can be seen that the first iteration is completed. Using the same iteration procedure, after three to five iterations usually, final form of functions w_x, ϕ_x, ψ_x and w_y, ϕ_y, ψ_y could be obtained. The integration is carried out analytically and the computational cost is negligible. We found the proposed Kantorovich method is effective in the analysis with lower frequencies at the flexural modes, in which the approximate solutions are almost exact in comparison with other methods such as the DQ Method. As the frequency increases, the selection of initial frequency and mode functions has to be careful to ensure the convergence. Finally, the checked and verified frequency and mode shape solutions are in good agreement with known results.

CONCLUSIONS

By using the known extended Kantorovich method, thickness-shear vibrations of anisotropic rectangular Mindlin plates are studied for the approximate solutions of vibration frequency and mode shapes. The results are compared with solutions from the finite element method showing good agreement. Such results are useful in the quick determination of quartz crystal resonator structural parameters.

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