EXTENDED KANTOROVICH METHOD FOR THREE-DIMENSIONAL PIEZOELASTICITY SOLUTION OF PIEZOELECTRIC PLATES

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Summary The extended Kantorovich method originally proposed by Kerr in 1968 for two-dimensional elasticity problems was recently generalized by the authors to the three-dimensional (3D) elasticity solution of elastic laminated plates. In this work, the method is further extended to obtain 3D piezoelasticity solutions of piezoelectric plates. It is shown that the solution converges very fast with just two to three terms in the trial function, and accurately predicts all mechanical and electrical variables, including the boundary layer effect.

INTRODUCTION

The boundary layer stress field in the vicinity of free or supported edges of laminated piezoelectric structures needs accurate prediction for ascertaining the integrity of the laminates as well as the sensory/actuator layers. The need for a general methodology for obtaining accurate analytical/semi-analytical solutions based on three-dimensional (3D) piezoelasticity in such cases has been felt for a long time [1]. Vel and Batra [2] presented a series solution using the Eshelby-Stroh formalism of 3D piezoelasticity for laminated piezoelectric plates with arbitrary boundary conditions. However, the interface continuity and the boundary conditions are satisfied in the sense of Fourier series, resulting in an infinite system of equations. The truncation in the series led to unwanted oscillations in the variations of transverse stresses at the boundaries. The powerful extended Kantorovich method (EKM) originally proposed by Kerr [3] in 1968, which has hitherto been used only for two-dimensional (2D) plate/shell problems [4] was recently extended by the authors [5, 6] to the 3D elasticity solution for elastic laminated plates. The solution was shown to yield very accurate results for all entities including the stress field near the boundaries for composite as well as sandwich plates under different boundary conditions. In this work, this method is further extended to obtain the 3D piezoelasticity solution for piezoelectric panels in cylindrical bending.

EXTENDED REISSNER-TYPE VARIATIONAL PRINCIPLE FOR 3D PIEZOELASTICITY

Consider a piezoelectric plate with span length a along x-direction, infinite length along y-direction and thickness h along z-direction. It is supported at two edges at $x = 0$ and a, with arbitrary boundary conditions, and is subjected to uniformly distributed pressure loads of p_1 and p_2 applied on the bottom and top surfaces, respectively. All entities are independent of y. The piezoelectric material exhibiting class mm2 symmetry and with poling along the thickness direction x_3, is orthorhombic with respect to the principal material directions x_1, x_2 and x_3. The mid-plane of the plate is chosen as the xy-plane. Using the 3D constitutive relations of a piezoelectric medium, and the strain-displacement and electric field-potential relations, the Reissner-type mixed variational principle for the piezoelectric case can be written as

$$\begin{aligned}\int_V &[\delta u(\tau_{xz,z} + \sigma_{x,x}) + \delta v(\tau_{yz,z} + \tau_{xy,x}) + \delta w(\sigma_{z,z} + \tau_{zx,x}) + \delta\phi(D_{x,x} + D_{z,z}) + \delta\sigma_x(p_{11}\sigma_x + p_{16}\tau_{xy} + p_{13}\sigma_z \\ &+p_{18}D_z - u_{,x}) - \delta\sigma_z(w_{,z} - p_{31}\sigma_x - p_{36}\tau_{xy} - p_{33}\sigma_z - p_{38}D_z) - \delta\tau_{yz}(v_{,z} - p_{44}\tau_{yz} - p_{45}\tau_{zx} - p_{47}D_x) \\ &-\delta\tau_{zx}(u_{,z} + w_{,x} - p_{54}\tau_{yz} - p_{55}\tau_{zx} - p_{57}D_x) + \delta\tau_{xy}(p_{61}\sigma_x + p_{66}\tau_{xy} + p_{63}\sigma_z + p_{68}D_z - v_{,x}) \\ &-\delta D_x(\phi_{,x} - p_{74}\tau_{yz} - p_{75}\tau_{zx} + p_{77}D_x) - \delta D_z(\phi_{,z} - p_{81}\sigma_x - p_{86}\tau_{xy} - p_{83}\sigma_z - p_{88}D_z)]dV = 0 \end{aligned} \qquad (1)$$

where a subscript comma denotes differentiation. V denotes the volume of the panel per unit width in the y direction. u, v, w are the displacements and ϕ is the electric potential. ε_i and γ_{ij} denote the normal and shearing strain components, σ_i and τ_{ij} denote the normal and shear stress components, D_i denotes the electric displacements, and E_i denotes the electric field components along x, y, z directions. p_{ij} are constants that are defined in terms of the electroelastic material properties. The boundary conditions at the top and bottom surfaces of the plate are:

$$\begin{aligned}&\text{at} \quad z = -h/2: \quad \sigma_z = -p_1, \quad \tau_{yz} = 0, \quad \tau_{zx} = 0, \quad \phi = \phi_1 \ \text{or} \ D_z = D_1 \\ &\text{at} \quad z = h/2: \quad \sigma_z = -p_2, \quad \tau_{yz} = 0, \quad \tau_{zx} = 0, \quad \phi = \phi_2 \ \text{or} \ D_z = D_2\end{aligned} \qquad (2)$$

The boundary conditions at the edges $x = 0$ and 1 can be arbitrary e.g. simply supported, clamped and free.

THE GENERALIZED EKM

The solution of the field variables, $\mathbf{X} = [\,u \quad v \quad w \quad \sigma_x \quad \sigma_z \quad \tau_{xy} \quad \tau_{yz} \quad \tau_{zx} \quad \phi \quad D_x \quad D_z\,]^{\mathrm{T}}$, is assumed in terms of an n-term series of the products of separable functions, which is superimposed with a known solution satisfying the non-homogenous boundary conditions: $X_l(x, z) = \sum_{i=1}^{n} f_l^i(x) g_l^i(z) + \delta_{l5}[p_a + z p_d] + \delta_{l9}[\phi_a + z\phi_d]$ for $l = 1, 2, \ldots, 11$, where δ_{ij} is Kronecker delta, $p_a = -(p_1 + p_2)/2$, $p_d = -(p_2 - p_1)/h$, $\phi_a = (\phi_1 + \phi_2)/2$ and $\phi_d = (\phi_2 - \phi_1)/h$. The repeated index l does not mean summation here. In the first iteration step, functions $f_l^i(x)$ are assumed, and functions $g_l^i(z)$ are partitioned into a column vector $\bar{\mathbf{G}}$ of those eight variables that appear in the boundary conditions given in Eq. (2)

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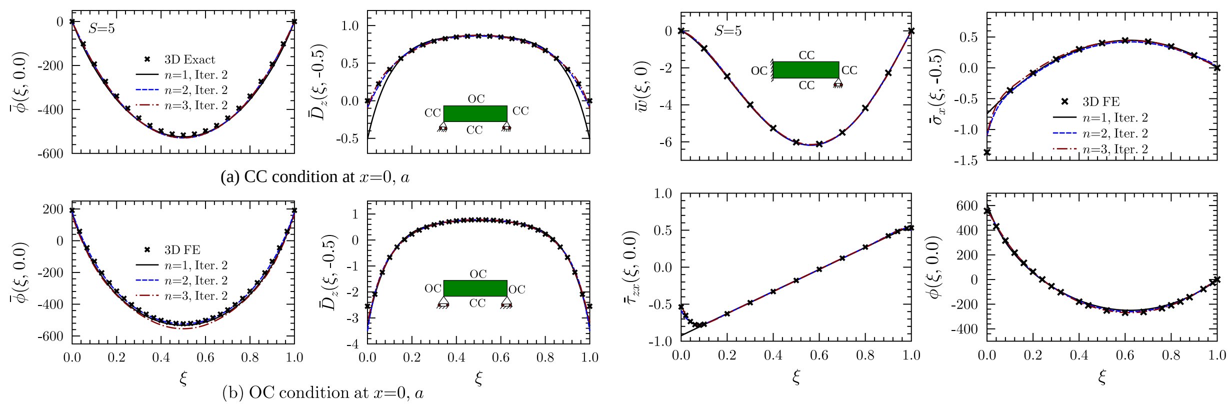
and a column vector $\hat{\mathbf{G}}$ consisting of the remaining three variables. Integrating Eq. (1) over x direction and considering that the variations δg_l^i are arbitrary result in the following set of $8n$ differential and $3n$ algebraic equations for δg_l^i:

$$\mathbf{M}\bar{\mathbf{G}}_{,z} = \bar{\mathbf{A}}\bar{\mathbf{G}} + \hat{\mathbf{A}}\hat{\mathbf{G}} + \bar{\mathbf{Q}}_\mathbf{p} + \bar{\mathbf{Q}}_\mathbf{e}; \qquad \mathbf{K}\hat{\mathbf{G}} = \tilde{\mathbf{A}}\bar{\mathbf{G}} + \tilde{\mathbf{Q}}_\mathbf{p} + \tilde{\mathbf{Q}}_\mathbf{e} \tag{3}$$

where $\mathbf{M}$, $\bar{\mathbf{A}}$, $\hat{\mathbf{A}}$, $\mathbf{K}$ and $\tilde{\mathbf{A}}$ are $8n \times 8n$, $8n \times 8n$, $8n \times 3n$, $3n \times 3n$ and $3n \times 8n$ matrices. $\bar{\mathbf{Q}}_\mathbf{p}$, $\bar{\mathbf{Q}}_\mathbf{e}$, $\tilde{\mathbf{Q}}_\mathbf{p}$, $\tilde{\mathbf{Q}}_\mathbf{e}$ are mechanical (p) and electric (e) load vectors. The elements of these matrices are not listed here for brevity. The system of the first order ordinary differential equations (ODEs) has constant coefficients and is solved analytically. In the second step, the solution for g_l^i obtained in the first step is taken as known a priori, and functions $f_l^i(x)$ are determined using a similar process. This completes one iteration, which is repeated until a prescribed level of convergence is achieved.

NUMERICAL RESULTS

Numerical results are presented for a piezoelectric panel made of PZT-5A (material properties are taken from Ref. [7]) subjected to uniform pressure loading at top surface ($p_1 = p_0$). The results are non-dimensionalized as: $(\bar{w}, \bar{\phi}) = (100w,\ 10^4\phi d_0 S^2)Y_0/p_0 h S^4$, $(\bar{\sigma}_x, \bar{\tau}_{zx}, \bar{D}_z) = (\sigma_x, S\tau_{zx}, D_z/d_0)/p_0 S^2$ where $S = a/h$, $Y_0 = 6.9$ GPa and $d_0 = 374.0 \times 10^{-12}$ CN^{-1}. Figure 1a shows the longitudinal variations of the induced electric potential $\bar{\phi}$ and electric displacement $\bar{D}_z$ of a thick simply supported panel with span-to-thickness ratio $S = 5$. In the figure, results for both close circuit (CC) and open circuit (OC) conditions at the edges are shown. The present results are compared with the available 3D exact piezoelasticity solution [7] for the CC case and with the 3D FE solution of ABAQUS for the OC case. For the simply-supported condition, an one-term ($n = 1$) solution itself gives very accurate prediction for the induced electric charge D_z at the top surface for the OC case, but it is not able do so for the CC case, yielding erroneous prediction of D_z near the edges. With just two terms in the trial function ($n = 2$), however, the EKM solution converges to the exact solution. Figure 1b shows the longitudinal variations of $\bar{w}$, $\bar{\sigma}_x$, $\bar{\tau}_{zx}$ and $\bar{\phi}$ of a clamped-simply supported panel. In this case also, while the single-term solution is unable to predict the sharp variations of $\bar{\sigma}_x$ and $\bar{\tau}_{zx}$ near the clamped support, the two-term solution predicts them very accurately including their through-thickness variations (not shown for brevity).



(a) Simply-supported panel (b) Clamped-simply supported panel

Figure 1. Longitudinal variations of deflection, stresses, electric potential and electric displacement for piezoelectric panels

CONCLUSIONS

The EKM is extended to 3D piezoelasticity solution for the cylindrical bending of piezoelectric plates for the first time, using a mixed variational principle. The solution converges very fast with just two terms in the trial function, establishing the elegance and efficacy of the method. The accuracy of the method is established by comparison with available 3D analytical and 3D FE solutions for displacements, stresses, electric potential as well as electric displacements. The present study will facilitate development of accurate semi-analytical solutions of many other unresolved problems in 3D piezoelasticity such as the free edge stresses in smart piezoelectric laminates under bending, tension and twisting.

References

[1] Mittelstedt C., Becker W.: Free-edge effects in composite laminates. *Appl. Mech. Rev.* **60**:217–244, 2007.

[2] Vel S.S., Batra R.C.: Cylindrical bending of laminated plates with distributed and segmented piezoelectric actuators/sensors. *AIAA J.* **38**:857–867, 2000.

[3] Kerr A.D.: An extension of the Kantorovich method. *Q. Appl. Math.* **4**:219–229, 1968.

[4] Aghdam M.M., Mohammadi M.: Bending analysis of thick orthotropic sector plates with various loading and boundary conditions. *Compos. Struct.* **88**:212–218, 2009.

[5] Kapuria S., Kumari P.: Extended Kantorovich method for three dimensional elasticity solution of laminated composite structures in cylindrical bending. *J. Appl. Mech* **78**:Article no. 061004, 2011.

[6] Kapuria S., Kumari P.: Multi-term extended Kantorovich method for three dimensional elasticity solution of laminated plates. *Comuunicated to J. Appl. Mech*, 2011.

[7] Kumari P., Nath J.K., Dumir P.C., Kapuria S.: 2D exact solutions for flat hybrid piezoelectric and magnetoelastic angle-ply panels under harmonic load *Smart Mater. Struct.* **16**:1651–1661 2007.